Supplement to "Bootstrapping Realized Volatility"

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This Appendix is organized as follows. First, we introduce some notation. Second, we provide auxiliary lemmas (and their proofs) used in deriving the cumulants expansions in Appendix A of the paper. Lastly, we prove Proposition 4.2 and part c) of Proposition 4.3, not included in Appendix B.

Notation

Recall that $\sigma_i^2 \equiv \int_{(i-1)h}^{ih} \sigma_u^2 du < \infty$, and for any $q > 0$, $\overline{\sigma_h^q} \equiv h^{1-q/2} \sum_{i=1}^{1/h} \left(\sigma_i^2\right)^{q/2} \equiv h^{1-q/2} \sum_{i=1}^{1/h} \sigma_i^q$, where $\sigma_i^q \equiv \left(\sigma_i^2\right)^{q/2}$. Note that in general $\overline{\sigma_h^q} \neq \overline{\sigma^q} \equiv \int_0^1 \sigma_u^q du$. We let $\sigma_{q,p} \equiv \frac{\overline{\sigma^q}}{\left(\overline{\sigma^p}\right)^{q/p}}$ for any $q, p > 0$. When $\overline{\sigma^q}$ is replaced with $\overline{\sigma_h^q}$ we write $\sigma_{q,p,h}$. Similarly, $R_{q,p} \equiv \frac{R_q}{\left(R_p\right)^{q/p}}$. We let $\mu_q = E\left|Z\right|^q$, where $Z \sim N(0,1)$ and $q > 0$, and note that $\mu_2 = 1$, $\mu_4 = 3$, $\mu_6 = 15$ and $\mu_8 = 105$. Since $\mu_2 = 1$, we can write $\overline{\sigma^2} = \mu_2 \overline{\sigma^2}$, which will be convenient for proving the results for the WB.

Write $T_h = S_h \left(\frac{\hat{V}}{V_h}\right)^{-1/2} = S_h \left(1 + \sqrt{h} U_h\right)^{-1/2}$, where

$$S_h \equiv \frac{\sqrt{h^{-1}}\left(R_2 - \mu_2 \overline{\sigma^2}\right)}{\sqrt{V_h}} \text{ and } U_h \equiv \frac{\sqrt{h^{-1}}\left(\hat{V} - V_h\right)}{V_h},$$

and $V_h = Var\left(\sqrt{h^{-1}} R_2\right) = \left(\mu_4 - \mu_2^2\right) \overline{\sigma_h^4}$. The proof of Lemma S.2 below relies heavily on the fact that

$$R_2 - \mu_2 \overline{\sigma^2} = \sum_{i=1}^{1/h} \left(r_i^2 - \mu_2 \sigma_i^2\right) \text{ and } \hat{V} - V_h = \frac{\mu_4 - \mu_2^2}{\mu_4} h^{-1} \sum_{i=1}^{1/h} \left(r_i^4 - \mu_4 \sigma_i^4\right),$$

where for any $q > 0$, $|r_i|^q - \mu_q \sigma_i^q$ are (conditionally on σ) independent with zero mean since $r_i = \sigma_i u_i$, where $u_i \sim$ i.i.d. $N(0,1)$.

Similarly, let $S_h^* \equiv \frac{\sqrt{h^{-1}}\left(R_2^* - E^*\left(R_2^*\right)\right)}{\sqrt{V^*}}$, $U_h^* \equiv \frac{\sqrt{h^{-1}}\left(\hat{V}^* - V^*\right)}{V^*}$, where $V^* = Var^*\left(h^{-1/2} R_2^*\right)$ and $\hat{V}^*$ is a consistent estimator of V^*. Then $T_h^* = S_h^* \left(1 + \sqrt{h} U_h^*\right)^{-1/2}$. For the i.i.d. bootstrap, $V^* = R_4 - R_2^2$ and $\hat{V}^* = R_4^* - R_2^{*2}$. For the WB, $V^* = \left(\mu_4^* - \mu_2^{*2}\right) R_4$ and $\hat{V}^* = \left(\frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\right) R_4^*$.

Finally, note that throughout we will use $\sum_{i \neq j \neq \cdots \neq k}$ to denote a sum where all indices differ, e.g. $\sum_{i \neq j \neq k} \equiv \sum_{i \neq j, i \neq k, j \neq k}$.

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Auxiliary Lemmas

Lemma S.1 *Let q, p and s be positive even integers. It follows that*

$$\sum_{i\neq j}^{1/h}\sigma_i^q\sigma_j^p = h^{-2+\frac{q+p}{2}}\left(\overline{\sigma_h^q}\ \overline{\sigma_h^p}-h\ \overline{\sigma_h^{q+p}}\right), \qquad (1)$$

$$\sum_{i\neq j\neq l}^{1/h}\sigma_i^q\sigma_j^p\sigma_l^s = h^{-3+\frac{q+p+s}{2}}\left(\overline{\sigma_h^q}\right)\left(\overline{\sigma_h^p}\right)\left(\overline{\sigma_h^s}\right)-h^{-2+\frac{q+p+s}{2}}\left(\overline{\sigma_h^{q+p}}\ \overline{\sigma_h^s}+\overline{\sigma_h^{q+s}}\ \overline{\sigma_h^p}+\overline{\sigma_h^q}\ \overline{\sigma_h^{p+s}}\right) \qquad (2)$$
$$+2h^{-1+\frac{q+p+s}{2}}\ \overline{\sigma_h^{q+p+s}}.$$

Lemma S.2 *Under Assumption H, conditionally on σ,*

a1) $E\left|r_i\right|^q=\mu_q\sigma_i^q$.

a2) $V_h\equiv Var\left(h^{-1/2}R_2\right)=\left(\mu_4-\mu_2^2\right)\overline{\sigma_h^4}$.

a3) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^3\right]=h^2\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\overline{\sigma_h^6}$.

a4) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^4\right]=3h^2\left(\mu_4-\mu_2^2\right)^2\left(\overline{\sigma_h^4}\right)^2+h^3\left(\mu_8-4\mu_2\mu_6+12\mu_2^2\mu_4-6\mu_2^4-3\mu_4^2\right)\overline{\sigma_h^8}$.

a5) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)\left(\hat{V}-V_h\right)\right]=h\dfrac{\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)}{\mu_4}\ \overline{\sigma_h^6}$.

a6) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^2\left(\hat{V}-V_h\right)\right]=h^2\dfrac{\mu_4-\mu_2^2}{\mu_4}\left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)\overline{\sigma_h^8}$.

a7) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^3\left(\hat{V}-V_h\right)\right]=3h^2\dfrac{\left(\mu_4-\mu_2^2\right)^2\left(\mu_6-\mu_2\mu_4\right)}{\mu_4}\overline{\sigma_h^4}\ \overline{\sigma_h^6}+O\left(h^3\right)$, *as* $h\to 0$.

a8) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^4\left(\hat{V}-V_h\right)\right]=h^3\dfrac{\mu_4-\mu_2^2}{\mu_4}\left[\begin{array}{c}4\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\left(\mu_6-\mu_2\mu_4\right)\left(\overline{\sigma_h^6}\right)^2\\+6\left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)\left(\mu_4-\mu_2^2\right)\overline{\sigma_h^4}\ \overline{\sigma_h^8}\end{array}\right]+$ $O\left(h^4\right)$, *as* $h\to 0$.

a9) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)\left(\hat{V}-V_h\right)^2\right] = \dfrac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}\left(\mu_{10}-2\mu_4\mu_6-\mu_2\mu_8+2\mu_2\mu_4^2\right)h^2\overline{\sigma_h^{10}} = O\left(h^2\right)$, *as* $h\to 0$.

a10) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^2\left(\hat{V}-V_h\right)^2\right]=h^2\dfrac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}\left(\left(\mu_4-\mu_2^2\right)\left(\mu_8-\mu_4^2\right)\overline{\sigma_h^4}\ \overline{\sigma_h^8}+2\left(\mu_6-\mu_2\mu_4\right)^2\left(\overline{\sigma_h^6}\right)^2\right)+$ $O\left(h^3\right)$, *as* $h\to 0$.

a11) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^3\left(\hat{V}-V_h\right)^2\right]=O\left(h^3\right)+O\left(h^4\right)$, *as* $h\to 0$.

a12) $E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^4\left(\hat{V}-V_h\right)^2\right]=h^3\dfrac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}\left[\begin{array}{c}3\left(\mu_4-\mu_2^2\right)^2\left(\mu_8-\mu_4^2\right)\left(\overline{\sigma_h^4}\right)^2\overline{\sigma_h^8}\\+12\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)^2\left(\overline{\sigma_h^6}\right)^2\overline{\sigma_h^4}\end{array}\right]+O\left(h^4\right)$, *as* $h\to 0$.

Lemma S.3 *Under Assumption H, conditionally on σ,*

$$\begin{aligned}
E\left(S_h\right) &= 0; \\
E\left(S_h^2\right) &= 1; \\
E\left(S_h^3\right) &= \sqrt{h}B_1\sigma_{6,4,h}; \\
E\left(S_h^4\right) &= 3+hB_2\sigma_{8,4,h}; \\
E\left(S_hU_h\right) &= A_1\sigma_{6,4,h}; \\
E\left(S_h^2U_h\right) &= \sqrt{h}\left(A_2\sigma_{8,4,h}\right);
\end{aligned}$$

and as $h\to 0$,

$$\begin{aligned}
E\left(S_h^3U_h\right) &= A_3\sigma_{6,4,h}+O\left(h\right); \\
E\left(S_h^4U_h\right) &= \sqrt{h}\left[D_1\sigma_{8,4,h}+D_2\sigma_{6,4,h}^2\right]+O\left(h^{3/2}\right); \\
E\left(S_hU_h^2\right) &= O\left(h^{1/2}\right);\ E\left(S_h^3U_h^2\right)=O\left(h^{1/2}\right); \\
E\left(S_h^2U_h^2\right) &= \left[C_1\sigma_{8,4,h}+C_2\sigma_{6,4,h}^2\right]+O\left(h\right); \\
E\left(S_h^4U_h^2\right) &= \left[E_1\sigma_{8,4,h}+E_2\sigma_{6,4,h}^2\right]+O\left(h\right).
\end{aligned}$$

The constants $A_1, A_2, B_1,\ B_2,$ and C_1 are as in Theorem A.1, and $A_3=3A_1,\ C_2=2A_1^2,\ D_1=6A_2,$ $D_2=4A_1B_1,\ E_1=3C_1,$ and $E_2=12A_1^2$.

Remark 1 *The WB analogue of Lemma S.2 replaces $\overline{\sigma_h^q}$ with R_q and μ_q with $\mu_q^*=E^*\left|\eta_i\right|^q$. The WB analogue of S.3 replaces $\sigma_{q,p,h}$ with $R_{q,p}$ (and μ_q with $\mu_q^*=E^*\left|\eta_i\right|^q$), yielding e.g. $E^*\left(S_h^{*3}\right)=\sqrt{h}\left(B_1^*R_{6,4,h}\right)$, where $B_1^*=\frac{\mu_6^*-3\mu_2^*\mu_4^*+2\mu_2^{*3}}{\left(\mu_4^*-\mu_2^{*2}\right)^{3/2}}$.*

Lemma S.7 below is the i.i.d. bootstrap analogue of Lemma S.3. The next results are auxiliary in proving Lemma S.7.

Lemma S.4 *Let $r_i^*\sim$ i.i.d. from $\{r_i : i=1,\ldots,1/h\}$. Under Assumption H, conditionally on σ, for any $q>0$ and for any $i=1,\ldots,1/h$,*

a1) $E^*\left(\left|r_i^*\right|^q\right)=h^{q/2}R_q$ *and* $E^*\left(R_q^*\right)=R_q=O_P\left(1\right).$

a2) $E^*\left[\left(r_i^{*2}-hR_2\right)^2\right]=h^2\left(R_4-R_2^2\right).$

a3) $E^*\left[\left(r_i^{*2}-hR_2\right)^3\right]=h^3\left(R_6-3R_4R_2+2R_2^3\right).$

a4) $E^*\left[\left(r_i^{*2}-hR_2\right)^4\right]=h^4\left(R_8-4R_6R_2+6R_4R_2^2-3R_2^4\right).$

a5) $E^*\left[\left(r_i^{*2}-hR_2\right)^5\right]=h^5\left(R_{10}-5R_8R_2+10R_6R_2^2-10R_4R_2^3+4R_2^5\right).$

a6) $E^*\left[\left(r_i^{*2}-hR_2\right)^6\right]=h^6\left(R_{12}-6R_{10}R_2+15R_8R_2^2-20R_6R_2^3+15R_4R_2^4-5R_2^6\right).$

a7) $E^*\left[\left(r_i^{*2}-hR_2\right)^q\right]=O_P\left(h^q\right)$, *for any $q\geq 7,$ as $h\to 0$.*

a8) $E^*\left[\left(r_i^{*4}-h^2R_4\right)^2\right]=h^4\left(R_8-R_4^2\right).$

a9) $E^{*}\left[\left(r_i^{*2}-hR_2\right)\left(r_i^{*4}-h^2R_4\right)\right]=h^3\left(R_6-R_4R_2\right).$

a10) $E^{*}\left[\left(r_i^{*2}-hR_2\right)^2\left(r_i^{*4}-h^2R_4\right)\right]=h^4\left(R_8-R_4^2-2R_6R_2+2R_4R_2^2\right).$

a11) $E^{*}\left[\left(r_i^{*2}-hR_2\right)^3\left(r_i^{*4}-h^2R_4\right)\right]=h^5\left(R_{10}-R_4R_6-3R_8R_2+3R_4^2R_2+3R_6R_2^2-3R_4R_2^3\right).$

a12) $E^{*}\left[\left(r_i^{*2}-hR_2\right)^4\left(r_i^{*4}-h^2R_4\right)\right]=h^6\left(\begin{array}{c}R_{12}-R_4R_8-4R_{10}R_2+4R_4R_6R_2+6R_8R_2^2\\-6R_4^2R_2^2-4R_6R_2^3+4R_4R_2^4\end{array}\right).$

a13) $E^{*}\left(\left(r_i^{*2}-hR_2\right)\left(r_i^{*4}-h^2R_4\right)^2\right)=h^5\left(R_{10}-2R_4R_6-R_8R_2+2R_4^2R_2\right).$

a14) *For any* $q,p>0$, $E^{*}\left[\left(r_i^{*2}-hR_2\right)^q\left(r_i^{*4}-h^2R_4\right)^p\right]=O_P\left(h^{q+2p}\right)$, *as* $h\to 0$.

Lemma S.5 *Let* $r_i^{*}\sim$ *i.i.d. from* $\{r_i : i=1,\ldots,1/h\}$. *Under Assumption H, conditionally on* σ,

a1) $V^{*}\equiv Var^{*}\left(h^{-1/2}R_2^{*}\right)=R_4-R_2^2.$

a2) $\hat{V}^{*}-V^{*}=R_4^{*}-R_4-\left[\left(R_2^{*}-R_2\right)^2+2R_2\left(R_2^{*}-R_2\right)\right].$

a3) $E^{*}\left[\left(R_2^{*}-R_2\right)^3\right]=h^2\left(R_6-3R_4R_2+2R_2^3\right).$

a4) $E^{*}\left[\left(R_2^{*}-R_2\right)^4\right]=h^2\left[3\left(R_4-R_2^2\right)^2\right]+h^3\left(R_8-4R_6R_2+12R_4R_2^2-6R_2^4-3R_4^2\right).$

a5) $E^{*}\left[\left(R_2^{*}-R_2\right)^5\right]=h^3\left[10\left(R_6-3R_4R_2+2R_2^3\right)\left(R_4-R_2^2\right)\right]+O_P\left(h^4\right)$, *as* $h\to 0$.

a6) $E^{*}\left[\left(R_2^{*}-R_2\right)^6\right]=h^3\left[15\left(R_4-R_2^2\right)^3\right]+O_P\left(h^4\right)$, *as* $h\to 0$.

a7) $E^{*}\left[\left(R_2^{*}-R_2\right)^q\right]=O_P\left(h^4\right)$ *for* $q=7,8$, *as* $h\to 0$.

a8) $E^{*}\left[\left(R_2^{*}-R_2\right)\left(R_4^{*}-R_4\right)\right]=h\left(R_6-R_4R_2\right).$

a9) $E^{*}\left[\left(R_2^{*}-R_2\right)^2\left(R_4^{*}-R_4\right)\right]=h^2\left(R_8-R_4^2-2R_6R_2+2R_4R_2^2\right).$

a10) $E^{*}\left[\left(R_2^{*}-R_2\right)^3\left(R_4^{*}-R_4\right)\right]=3h^2\left(R_6-R_4R_2\right)\left(R_4-R_2^2\right)+O_P\left(h^3\right)$, *as* $h\to 0$.

a11) $E^{*}\left[\left(R_2^{*}-R_2\right)^4\left(R_4^{*}-R_4\right)\right]=h^3\left[\begin{array}{c}4\left(R_6-3R_4R_2+2R_2^3\right)\left(R_6-R_4R_2\right)\\+6\left(R_4-R_2^2\right)\left(R_8-R_4^2-2R_6R_2+2R_4R_2^2\right)\end{array}\right]+O_P\left(h^4\right)$, *as* $h\to 0$.

a12) $E^{*}\left[\left(R_2^{*}-R_2\right)^5\left(R_4^{*}-R_4\right)\right]=h^3\left[15\left(R_4-R_2^2\right)^2\left(R_6-R_4R_2\right)\right]+O_P\left(h^4\right)$, *as* $h\to 0$.

a13) $E^{*}\left[\left(R_2^{*}-R_2\right)^6\left(R_4^{*}-R_4\right)\right]=O_P\left(h^4\right)$, *as* $h\to 0$.

a14) $E^{*}\left[\left(R_2^{*}-R_2\right)\left(R_4^{*}-R_4\right)^2\right]=h^2\left(R_{10}-2R_4R_6-R_8R_2+2R_4^2R_2\right).$

a15) $E^{*}\left[\left(R_2^{*}-R_2\right)^2\left(R_4^{*}-R_4\right)^2\right]=h^2\left[\left(R_4-R_2^2\right)\left(R_8-R_4^2\right)+2\left(R_6-R_4R_2\right)^2\right]+O_P\left(h^3\right)$, *as* $h\to 0$.

a16) $E^{*}\left[\left(R_2^{*}-R_2\right)^3\left(R_4^{*}-R_4\right)^2\right]=O_P\left(h^3\right)$, *as* $h\to 0$.

a17) $E^*\left[(R_2^*-R_2)^4(R_4^*-R_4)^2\right]=h^3\left[\begin{array}{c}3\left(R_4-R_2^2\right)^2\left(R_8-R_4^2\right)\\+12\left(R_6-R_4R_2\right)^2\left(R_4-R_2^2\right)\end{array}\right]+O_P\left(h^4\right),$ *as* $h\to 0.$

Lemma S.6 *Let* $r_i^*\sim$ *i.i.d. from* $\{r_i:i=1,\ldots,1/h\}$. *Under Assumption H, conditionally on* σ, *as* $h\to 0$,

a1) $E^*\left[(R_2^*-R_2)\left(\hat{V}^*-V^*\right)\right]=h\left(R_6-3R_4R_2+2R_2^3\right)+O_P\left(h^2\right).$

a2) $E^*\left[(R_2^*-R_2)^2\left(\hat{V}^*-V^*\right)\right]=h^2\left[\begin{array}{c}\left(R_8-R_4^2-2R_6R_2+2R_4R_2^2\right)-3\left(R_4-R_2^2\right)^2\\-2R_2\left(R_6-3R_4R_2+2R_2^3\right)\end{array}\right]+O_P\left(h^3\right).$

a3) $E^*\left[(R_2^*-R_2)^3\left(\hat{V}^*-V^*\right)\right]=h^2\left[3\left(R_4-R_2^2\right)\left(R_6-3R_4R_2+2R_2^3\right)\right]+O_P\left(h^3\right).$

a4)

$$\begin{aligned}E^*\left[(R_2^*-R_2)^4\left(\hat{V}^*-V^*\right)\right] &= h^3\left[\begin{array}{c}4\left(R_6-3R_4R_2+2R_2^3\right)\left(R_6-R_4R_2\right)\\+6\left(R_4-R_2^2\right)\left(R_8-R_4^2-2R_6R_2+2R_4R_2^2\right)\end{array}\right]\\&\quad -h^3\left[15\left(R_4-R_2^2\right)^3\right]\\&\quad -h^3\left[20R_2\left(R_6-3R_4R_2+2R_2^3\right)\left(R_4-R_2^2\right)\right]+O_P\left(h^4\right).\end{aligned}$$

a5) $E^*\left[(R_2^*-R_2)\left(\hat{V}^*-V^*\right)^2\right]=O_P\left(h^2\right).$

a6) $E^*\left[(R_2^*-R_2)^2\left(\hat{V}^*-V^*\right)^2\right]=h^2\left[\begin{array}{c}\left(R_4-R_2^2\right)\left(R_8-R_4^2\right)+2\left(R_6-R_4R_2\right)^2\\-12\left(R_6-R_4R_2\right)\left(R_4-R_2^2\right)\left(R_2\right)\\+4\left(R_2\right)^2\left[3\left(R_4-R_2^2\right)^2\right]\end{array}\right]+O_P\left(h^3\right).$

a7) $E^*\left[(R_2^*-R_2)^3\left(\hat{V}^*-V^*\right)^2\right]=O_P\left(h^3\right).$

a8)

$$\begin{aligned}E^*\left[(R_2^*-R_2)^4\left(\hat{V}^*-V^*\right)^2\right] &= h^3\left[3\left(R_4-R_2^2\right)^2\left(R_8-R_4^2\right)+12\left(R_6-R_4R_2\right)^2\left(R_4-R_2^2\right)\right]\\&\quad -h^3\left[60\left(R_4-R_2^2\right)^2\left(R_6-R_4R_2\right)\left(R_2\right)\right]\\&\quad +h^3\left[60\left(R_4-R_2^2\right)^3\left(R_2\right)^2\right]+O_P\left(h^4\right).\end{aligned}$$

Lemma S.7 *Let* $r_i^*\sim$ *i.i.d. from* $\{r_i:i=1,\ldots,1/h\}$. *Under Assumption H, conditionally on* σ,

$$\begin{aligned}E^*\left(S_h^*\right) &= 0;\\E^*\left(S_h^{*2}\right) &= 1;\\E^*\left(S_h^{*3}\right) &= \sqrt{h}\tilde{B}_1;\\E^*\left(S_h^{*4}\right) &= 3+h\tilde{B}_2;\end{aligned}$$

and as $h \to 0$,

$$\begin{aligned}
E^*\left(S_h^* U_h^*\right) &= \tilde{A}_1 + O_P\left(h\right); \\
E^*\left(S_h^{*2} U_h^*\right) &= \sqrt{h}\tilde{A}_2 + O_P\left(h^{3/2}\right); \\
E^*\left(S_h^{*3} U_h^*\right) &= \tilde{A}_3 + O_P\left(h\right); \\
E^*\left(S_h^{*4} U_h^*\right) &= \sqrt{h}\tilde{D} + O_P\left(h^{3/2}\right); \\
E^*\left(S_h^{*} U_h^{*2}\right) &= O_P\left(h^{1/2}\right); \ E^*\left(S_h^{*3} U_h^{*2}\right) = O_P\left(h^{1/2}\right); \\
E^*\left(S_h^{*2} U_h^{*2}\right) &= \tilde{C} + O_P\left(h\right); \ \text{and } E^*\left(S_h^{*4} U_h^{*2}\right) = \tilde{E} + O_P\left(h\right).
\end{aligned}$$

The bootstrap constants $\tilde{A}_1, \tilde{A}_2, \tilde{B}_2, \tilde{C}, \tilde{D}$ *and* $\tilde{E}$ *are as in Theorem A.2, and* $\tilde{A}_3$ *and* $\tilde{B}_1$ *are such that* $\tilde{A}_3 = 3\tilde{A}_1$ *and* $\tilde{B}_1 = \tilde{A}_1$.

Proof of Lemmas S.1–S.7

Proof of Lemma S.1. For (1), note that

$$\begin{aligned}
\sum_{i \neq j}^{1/h} \sigma_i^q \sigma_j^p &= \left(\sum_{i=1}^{1/h} \sigma_i^q\right)\left(\sum_{j=1}^{1/h} \sigma_j^p\right) - \left(\sum_{i=1}^{1/h} \sigma_i^{q+p}\right) \\
&= h^{-1+\frac{q}{2}}\left(h^{1-\frac{q}{2}}\sum_{i=1}^{1/h} \sigma_i^q\right) h^{-1+\frac{p}{2}}\left(h^{1-\frac{p}{2}}\sum_{j=1}^{1/h} \sigma_j^p\right) - h^{-1+\frac{q+p}{2}}\left(h^{1-\frac{q+p}{2}}\sum_{i=1}^{1/h} \sigma_i^{q+p}\right) \\
&= h^{-2+\frac{q+p}{2}}\left(\overline{\sigma_h^q}\ \overline{\sigma_h^p} - h\ \overline{\sigma_h^{q+p}}\right).
\end{aligned}$$

For (2), note that

$$\sum_{i \neq j \neq k} \sigma_i^q \sigma_j^p \sigma_k^s = \left(\sum_{i=1}^{1/h} \sigma_i^q\right)\left(\sum_{j=1}^{1/h} \sigma_j^p\right)\left(\sum_{k=1}^{1/h} \sigma_k^s\right) - \sum_{i=1}^{1/h} \sigma_i^{q+p+s} - \sum_{i \neq j} \sigma_i^{q+p}\sigma_j^s - \sum_{i \neq j} \sigma_i^{q+s}\sigma_j^p - \sum_{i \neq j} \sigma_i^q \sigma_j^{p+s},$$

and then proceed as for (1).

Proof of Lemma S.2. a1) follows from $r_i = \sigma_i u_i$, where $u_i \sim$ i.i.d. $N(0,1)$. For a2), note that $R_2 = \sum_{i=1}^{1/h} r_i^2$, where r_i^2 is (conditional on σ) independent with $Var\left(r_i^2\right) = E\left(r_i^4\right) - \left(E\left(r_i^2\right)\right)^2 = \mu_4\sigma_i^4 - \left(\mu_2\sigma_i^2\right)^2 = \left(\mu_4 - \mu_2^2\right)\sigma_i^4$, with $\sigma_i^4 \equiv \left(\sigma_i^2\right)^2$. To prove the remaining results we use the multinomial formula to compute the coefficients in the expansions. In particular, we have that

$$(a_1 + a_2 + \cdots + a_d)^n = \sum_{\substack{n_1, n_2, \ldots, n_d \geq 0 \\ n_1 + \cdots + n_d = n}} \frac{n!}{n_1! n_2! \cdots n_d!} a_1^{n_1} a_2^{n_2} \cdots a_d^{n_d}.$$

Proof of a3): Write

$$I_1 \equiv E\left[\left(R_2 - \mu_2\overline{\sigma^2}\right)^3\right] = E\left[\sum_{i=1}^{1/h}\sum_{j=1}^{1/h}\sum_{k=1}^{1/h}\left(r_i^2 - \mu_2\sigma_i^2\right)\left(r_j^2 - \mu_2\sigma_j^2\right)\left(r_k^2 - \mu_2\sigma_k^2\right)\right].$$

The only non zero contribution to I_1 is when $i=j=k$, in which case we get $E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]=\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\sigma_i^6$ and $I_1=h^2\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\overline{\sigma_h^6}$, proving a3). *Proof of a4):* Using the independence and zero mean property of $\left\{r_i^2-\mu_2\sigma_i^2\right\}$, we have that

$$\begin{aligned}
E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)^4\right] &= \sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\right]+3\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\right]\\
&= E\left[\left(u_i^2-\mu_2\right)^4\right]\sum_{i=1}^{1/h}\sigma_i^8+3\left(E\left[\left(u_i^2-\mu_2\right)^2\right]\right)^2\sum_{i\neq j}^{1/h}\sigma_i^4\sigma_j^4\\
&= \left(\mu_8-3\mu_2^4+6\mu_2^2\mu_4-4\mu_2\mu_6\right)h^3\overline{\sigma_h^8}+3\left(\mu_4-\mu_2^2\right)^2\left[h^2\left(\left(\overline{\sigma_h^4}\right)^2-h\left(\overline{\sigma_h^8}\right)\right)\right]\\
&= 3h^2\left(\mu_4-\mu_2^2\right)^2\left(\overline{\sigma_h^4}\right)^2+h^3\left(\mu_8-4\mu_2\mu_6+12\mu_4\mu_2^2-6\mu_2^4-3\mu_4^2\right)\left(\overline{\sigma_h^8}\right),
\end{aligned}$$

where we have made use of Lemma S.1 and of the following results:

$$\begin{aligned}
E\left[\left(u_i^2-\mu_2\right)^4\right] &= E\left[u_i^8-4u_i^2\mu_2^3+6u_i^4\mu_2^2-4u_i^6\mu_2+\mu_2^4\right]=\mu_8-3\mu_2^4+6\mu_2^2\mu_4-4\mu_2\mu_6.\\
E\left[\left(u_i^2-\mu_2\right)^2\right] &= E\left[u_i^4-2u_i^2\mu_2+\mu_2^2\right]=\mu_4-\mu_2^2.
\end{aligned}$$

Proof of a5):

$$\begin{aligned}
E\left[\left(R_2-\mu_2\overline{\sigma^2}\right)\left(\hat{V}-V_h\right)\right] &= \frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}\sum_{i=1}^{1/h}E\left(r_i^6-r_i^2\mu_4\sigma_i^4-\mu_2\sigma_i^2r_i^4+\mu_2\mu_4\sigma_i^6\right)\\
&= \frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}h^2\left(\mu_6-\mu_2\mu_4\right)\overline{\sigma_h^6}=h\frac{\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)}{\mu_4}\overline{\sigma_h^6}.
\end{aligned}$$

Proof of a6):

$$\begin{aligned}
E\left(\left(R_2-\mu_2\overline{\sigma^2}\right)^2\left(\hat{V}-V_h\right)\right) &= \frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}\sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)\right]\\
&= h^2\frac{\left(\mu_4-\mu_2^2\right)\left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)}{\mu_4}\overline{\sigma_h^8}.
\end{aligned}$$

Proof of a7): Write $E\left(\left(R_2-\mu_2\overline{\sigma^2}\right)^3\left(\hat{V}-V_h\right)\right)=\frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}I_2$, where by the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$,

$$\begin{aligned}
I_2 &= \sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\left(r_i^4-\mu_4\sigma_i^4\right)\right]+3\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right]\\
&= M_1\sum_{i=1}^{1/h}\sigma_i^{10}+3E\left[\left(u_i^2-\mu_2\right)^2\right]E\left[\left(u_j^2-\mu_2\right)\left(u_j^4-\mu_4\right)\right]\sum_{i\neq j}^{1/h}\sigma_i^4\sigma_j^6\\
&= 3h^3\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)\overline{\sigma_h^4}\ \overline{\sigma_h^6}+O\left(h^4\right),
\end{aligned}$$

given Lemma S.1, the fact that $\overline{\sigma_h^{10}}=O(1)$ under our assumptions, and where $M_1=E\left[\left(u_i^2-\mu_2\right)^3\left(u_i^4-\mu_4\right)\right]$ is a constant, and $E\left[\left(u_i^2-\mu_2\right)^2\right]=\mu_4-\mu_2^2$ and $E\left[\left(u_j^2-\mu_2\right)\left(u_i^4-\mu_4\right)\right]=\mu_6-\mu_2\mu_4$.

Proof of a8): Write $E\left(\left(R_2-\mu_2\overline{\sigma^2}\right)^4\left(\hat{V}-V_h\right)\right)=\frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}I_3$, where by the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$, and Lemma S.1,

$$\begin{aligned}
I_3 &= \sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\left(r_i^4-\mu_4\sigma_i^4\right)\right]+4\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right]\\
&\quad +6\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\left(r_j^4-\mu_4\sigma_j^4\right)\right]\\
&= M_1h^5\overline{\sigma_h^{12}}+4M_2\left(h^4\left(\overline{\sigma_h^6}\right)^2-h^5\overline{\sigma_h^{12}}\right)+6M_3\left(h^4\overline{\sigma_h^4}\ \overline{\sigma_h^8}-h^5\overline{\sigma_h^{12}}\right)\\
&= h^4\left[4M_2\left(\overline{\sigma_h^6}\right)^2+6M_3\overline{\sigma_h^4}\ \overline{\sigma_h^8}\right]+O\left(h^5\right),
\end{aligned}$$

where $M_1\equiv E\left[\left(u_i^2-\mu_2\right)^4\left(u_i^4-\mu_4\right)\right]$, $M_2\equiv E\left[\left(u_i^2-\mu_2\right)^3\right]E\left[\left(u_j^2-\mu_2\right)\left(u_j^4-\mu_4\right)\right]$
$=\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\left(\mu_6-\mu_2\mu_4\right)$ and $M_3\equiv E\left[\left(u_i^2-\mu_2\right)^2\right]E\left[\left(u_i^2-\mu_2\right)^2\left(u_i^4-\mu_4\right)\right]$
$=\left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)\left(\mu_4-\mu_2^2\right)$, and given the fact that $\overline{\sigma_h^q}=O(1)$ under our assumptions.

Proof of a9): Write $E\left(\left(R_2-\mu_2\overline{\sigma^2}\right)\left(\hat{V}-V_h\right)^2\right)=\frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}\sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)\left(r_i^4-\mu_4\sigma_i^4\right)^2\right]=O\left(h^2\right)$.

Proof of a10): Write $E\left(\left(R_2-\mu_2\overline{\sigma^2}\right)^2\left(\hat{V}-V_h\right)^2\right)=\frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}I_4$, where by the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$,

$$\begin{aligned}
I_4 &= \sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)^2\right]+\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^4-\mu_4\sigma_j^4\right)^2\right]\\
&\quad +2\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right]\\
&= D_1h^5\overline{\sigma_h^{12}}+D_2\left(h^4\overline{\sigma_h^4}\ \overline{\sigma_h^8}-h^5\overline{\sigma_h^{12}}\right)+2D_3\left(h^4\left(\overline{\sigma_h^6}\right)^2-h^5\overline{\sigma_h^{12}}\right)\\
&= h^4\left(D_2\overline{\sigma_h^4}\ \overline{\sigma_h^8}+2D_3\left(\overline{\sigma_h^6}\right)^2\right)+O\left(h^5\right),
\end{aligned}$$

given Lemma S.1, and where $D_1=E\left[\left(u_i^2-\mu_2\right)^2\left(u_i^4-\mu_4\right)^2\right]$, $D_2=E\left[\left(u_i^2-\mu_2\right)^2\right]E\left[\left(u_j^4-\mu_4\right)^2\right]=\left(\mu_4-\mu_2^2\right)\left(\mu_8-\mu_4^2\right)$ and $D_3=\left[E\left(\left(u_i^2-\mu_2\right)\left(u_i^4-\mu_4\right)\right)\right]^2=\left(\mu_6-\mu_2\mu_4\right)^2$.

Proof of a11): Write $E\left(\left(R_2-\mu_2\overline{\sigma^2}\right)^3\left(\hat{V}-V_h\right)^2\right)=\frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}I_5$, with

$$\begin{aligned}
I_5 &= \sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\left(r_i^4-\mu_4\sigma_i^4\right)^2\right]+\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]E\left[\left(r_j^4-\mu_4\sigma_j^4\right)^2\right]\\
&\quad+3\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)^2\right]\\
&\quad+6\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right]\\
&= K_1h^6\overline{\sigma_h^{14}}+K_2\left(h^5\overline{\sigma_h^6}\ \overline{\sigma_h^8}-h^6\overline{\sigma_h^{14}}\right)+3K_3\left(h^5\overline{\sigma_h^4}\ \overline{\sigma_h^{10}}-h^6\overline{\sigma_h^{14}}\right)+6K_4\left(h^5\overline{\sigma_h^8}\ \overline{\sigma_h^6}-h^6\overline{\sigma_h^{14}}\right)\\
&= h^5\left(K_2\overline{\sigma_h^6}\ \overline{\sigma_h^8}+3K_3\overline{\sigma_h^4}\ \overline{\sigma_h^{10}}+6K_4\overline{\sigma_h^8}\ \overline{\sigma_h^6}\right)+h^6\left(K_1-K_2-3K_3-6K_4\right)\overline{\sigma_h^{14}},
\end{aligned}$$

where we have used the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$, Lemma S.1, and where K_1 through K_4 are constants depending on μ_q. Since $\overline{\sigma_h^q}=O(1)$, the result follows.

Proof of a12): Write $E\left(\left(R_2-\mu_2\overline{\sigma^2}\right)^4\left(\hat{V}-V_h\right)^2\right)=\frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}I_6$, with

$$\begin{aligned}
I_6 &= \sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\left(r_i^4-\mu_4\sigma_i^4\right)^2\right]+\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\right]E\left[\left(r_j^4-\mu_4\sigma_j^4\right)^2\right]\\
&\quad+8\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right]\\
&\quad+6\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\right]\\
&\quad+4\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)^2\right]\\
&\quad+6\sum_{i\neq j}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\left(r_j^4-\mu_4\sigma_j^4\right)\right]\\
&\quad+3\sum_{i\neq j\neq k}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\right]E\left[\left(r_k^4-\mu_4\sigma_k^4\right)^2\right]\\
&\quad+12\sum_{i\neq j\neq k}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right]E\left[\left(r_k^2-\mu_2\sigma_k^2\right)\left(r_k^4-\mu_4\sigma_k^4\right)\right]\\
&= h^5\left[3J_7\left(\overline{\sigma_h^4}\right)^2\overline{\sigma_h^8}+12J_8\left(\overline{\sigma_h^6}\right)^2\overline{\sigma_h^4}\right]+O\left(h^6\right)+O\left(h^7\right),
\end{aligned}$$

given the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$, Lemma S.1, and where $J_7=\left(E\left[\left(u_i^2-\mu_2\right)^2\right]\right)^2E\left[\left(u_i^4-\mu_4\right)^2\right]=\left(\mu_4-\mu_2^2\right)^2\left(\mu_8-\mu_4^2\right)$ and $J_8=E\left[\left(u_i^2-\mu_2\right)^2\right]\left(E\left[\left(u_i^2-\mu_2\right)\left(u_i^4-\mu_4\right)\right]\right)^2=\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)^2.$

Proof of Lemma S.3. The first two results are obvious given S_h. The remaining results follow from the definition of S_h and Lemma S.2.

Proof of Lemma S.4. Part a1) follows from the properties of the i.i.d. bootstrap. The remaining results follow from a1), given the binomial expansions. Note in particular that since $R_q = O_P(1)$, it follows that $E^*\left[\left(r_i^{*2} - hR_2\right)^q\right] = O_P(h^q)$. For instance, for a2), $E^*\left[\left(r_i^{*2} - hR_2\right)^2\right] = E^*\left(r_i^{*4} - 2r_i^{*2}hR_2 + (hR_2)^2\right) = h^2\left(R_4 - R_2^2\right)$. The other results follows similarly.

Proof of Lemma S.5. For a1), since r_i^* are i.i.d. from $\{r_i : i = 1, \ldots, 1/h\}$, it follows that

$$V^* = h^{-1}Var^*\left(\sum_{i=1}^{1/h} r_i^{*2}\right) = h^{-1}\sum_{i=1}^{1/h} Var^*\left(r_i^{*2}\right) = h^{-2}Var^*\left(r_1^{*2}\right).$$

But $Var^*\left(r_1^{*2}\right) = E^*\left(r_1^{*4}\right) - \left(E^*\left(r_1^{*2}\right)\right)^2 = h^2R_4 - (hR_2)^2$. Thus, $V^* = R_4 - R_2^2$. Part a2) follows because $V^* = R_4 - R_2^2$ and $\hat{V}^* = R_4^* - R_2^{*2}$. For the remaining of the proof, note that $\sum_{i\neq j}^{1/h} 1 = h^{-2} - h^{-1}$, $\sum_{i\neq j\neq k} 1 = h^{-3} + 2h^{-1} - 3h^{-2}$, and $\sum_{i\neq j\neq k\neq m}^{1/h} 1 = h^{-4} - 6h^{-3} + 11h^{-2} - 6h^{-1}$. In addition, note that

$$R_2^* - R_2 = \sum_{i=1}^{1/h}\left(r_i^{*2} - hR_2\right) \quad \text{and} \quad R_4^* - R_4 = h^{-1}\sum_{i=1}^{1/h}\left(r_i^{*4} - h^2R_4\right),$$

where for any $q > 0$ $\left\{|r_i^*|^q - h^{q/2}R_q\right\}$ are (conditionally on the sample) i.i.d. with zero mean, and $R_q = O_P(1)$. Using this independence property, we evaluate the bootstrap expectations of the sums of products and cross products of $|r_i^*|^q - h^{q/2}R_q$ by relying on Lemma S.4 to compute the appropriate bootstrap moments of products and cross products of $|r_i^*|^q - h^{q/2}R_q$. We proceed as in the proof of Lemma S.2 and use the multinomial expansions to compute the number of coefficients in each sum.

Proof of Lemma S.6. Using part a2) of Lemma S.5, for $q = 1, \ldots, 4$, we can write

$$\begin{aligned} E^*\left[(R_2^* - R_2)^q\left(\hat{V}^* - V^*\right)\right] &= E^*\left[(R_2^* - R_2)^q(R_4^* - R_4)\right] - E^*\left[(R_2^* - R_2)^{2+q}\right] \\ &\quad -2(R_2)E^*\left[(R_2^* - R_2)^{1+q}\right] \\ &\equiv I_1^q - I_2^q - I_3^q. \end{aligned} \tag{3}$$

Similarly, for $q = 1, \ldots, 4$, note that

$$\begin{aligned} E^*\left[(R_2^* - R_2)^q\left(\hat{V}^* - V^*\right)^2\right] &= E^*\left[(R_2^* - R_2)^q(R_4^* - R_4)^2\right] - 2E^*\left[(R_2^* - R_2)^{2+q}(R_4^* - R_4)\right] \\ &\quad -4(R_2)E^*\left[(R_2^* - R_2)^{1+q}(R_4^* - R_4)\right] + E^*\left[(R_2^* - R_2)^{4+q}\right] \\ &\quad +4(R_2)E^*\left[(R_2^* - R_2)^{3+q}\right] + 4(R_2)^2E^*\left[(R_2^* - R_2)^{2+q}\right] \end{aligned}$$

For a1), set $q = 1$ in (3). We have that

$$\begin{aligned} I_1^1 &= E^*\left[(R_2^* - R_2)(R_4^* - R_4)\right] = h(R_6 - R_4R_2) \\ I_2^1 &= E^*\left[(R_2^* - R_2)^3\right] = h^2\left(R_6 - 3R_4R_2 + 2R_2^3\right) \\ I_3^1 &= 2R_2E^*\left[(R_2^* - R_2)^2\right] = 2R_2\left[h\left(R_4 - R_2^2\right)\right], \end{aligned}$$

by Lemma S.5. a8), a3), a1), respectively. Thus

$$\begin{aligned} E^*\left[(R_2^* - R_2)\left(\hat{V}^* - V^*\right)\right] &= h\left[(R_6 - R_4R_2) - 2R_2\left(R_4 - R_2^2\right)\right] - h^2\left(R_6 - 3R_4R_2 + 2R_2^3\right) \\ &= h\left(R_6 - 3R_4R_2 + 2R_2^3\right) + O_P\left(h^2\right). \end{aligned}$$

The remaining results follow similarly.
Proof of Lemma S.7. The proof follows the proof of Lemma S.3, given the definition of V^* and given Lemmas S.5 and S.6.

Proof of Propositions 4.2 and 4.3.c) in Section 4

Proof of Proposition 4.2. When $g(z) = \log z$, we have that

$$q_{1,\log}(x) = \frac{\sqrt{2}}{3}\frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}}(2x^2+1) + \frac{(\overline{\sigma^4})^{1/2}}{\sqrt{2}}\frac{g''_{\log}(\overline{\sigma^2})}{g'_{\log}(\overline{\sigma^2})} = \frac{1}{\sqrt{2}}\frac{(\overline{\sigma^4})^{1/2}}{\overline{\sigma^2}}\left[x^2\left(\frac{4}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2} - 1\right) + \frac{2}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2}\right],$$

whereas for $g(z) = z$, $g''(z) = 0$ and

$$q_1(x) = \frac{\sqrt{2}}{3}\frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}}(2x^2+1) = \frac{1}{\sqrt{2}}\frac{(\overline{\sigma^4})^{1/2}}{\overline{\sigma^2}}\left[x^2\left(\frac{4}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2}\right) + \frac{2}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2}\right].$$

Since $(\overline{\sigma^4})^2 \leq \overline{\sigma^6}\,\overline{\sigma^2}$ by the Cauchy-Schwartz inequality, it follows that $\frac{1}{3} \leq \frac{4}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2} - 1 < \frac{4}{3}\frac{\overline{\sigma^6}\,\overline{\sigma^2}}{(\overline{\sigma^4})^2}$, which implies that $q_{1,\log}(x) < q_1(x)$ for x fixed and non-zero. When $x = 0$, it follows trivially that $q_{1,\log}(0) = q_1(0)$.
Proof of Proposition 4.3.c). Define $C = \frac{4\overline{\sigma^6}}{\sqrt{2}(\overline{\sigma^4})^{3/2}}$ and $C^* = \frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}}$, and note that $C > 0$. It suffices to prove that $|C - C^*| \leq |C|$, which in turn is equivalent to proving $0 \leq C^* \leq 2C$. Next we show that $C^* \geq 0$. The Jensen's inequality implies that $\overline{\sigma^4} \geq (\overline{\sigma^2})^2$, and since $\overline{\sigma^4} > 0$, it follows that the denominator of C^* is positive. For the numerator of C^*, note we can write

$$15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3 \geq 15\overline{\sigma^6} - 9(\overline{\sigma^4})^{3/2} + 2(\overline{\sigma^2})^3 \geq 9(\overline{(\sigma^4)^{3/2}} - (\overline{\sigma^4})^{3/2}) + 6\overline{\sigma^6} + 2(\overline{\sigma^2})^3,$$

using $-(\overline{\sigma^2})^2 \geq -\overline{\sigma^4}$. Since the function $\psi(x) = x^{3/2}$ for $x > 0$ is convex, we have that $\overline{(\sigma^4)^{3/2}} - (\overline{\sigma^4})^{3/2} \geq 0$, which implies $15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3 \geq 6\overline{\sigma^6} + 2(\overline{\sigma^2})^3 > 0$, proving that the numerator of C^* is also positive. Next we prove $\frac{C^*}{C} \leq 2$. We can write

$$\frac{C^*}{C} = \frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3}{8\overline{\sigma^6}}\frac{2\sqrt{2}(\overline{\sigma^4})^{3/2}}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}} \equiv C_1 \times C_2.$$

We show that $C_1 \leq 2$ and $C_2 \leq 1$. First, note that

$$\frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3}{8\overline{\sigma^6}} \leq 2 \iff 15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3 \leq 16\overline{\sigma^6} \iff 0 \leq \overline{\sigma^6} + 7\overline{\sigma^4}\,\overline{\sigma^2} + 2\overline{\sigma^2}\left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right),$$

which proves the result since $\left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right) \geq 0$ and $0 \leq \overline{\sigma^6} + 7\overline{\sigma^4}\,\overline{\sigma^2}$. Finally, we have that

$$\frac{2\sqrt{2}(\overline{\sigma^4})^{3/2}}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}} \leq 1 \iff 8(\overline{\sigma^4})^3 \leq \left(3\overline{\sigma^4} - (\overline{\sigma^2})^2\right)^3 \iff 8(\overline{\sigma^4})^3 \leq \left(2\overline{\sigma^4} + \left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right)\right)^3,$$

which holds true since $\left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right) \geq 0$.