THE MATHEMATICAL FOUNDATIONS OF QUANTUM MECHANICS BY HILBERT AND VON NEUMANN

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1. INTRODUCTION

Mathematical physics can hardly be denied the status of a cross-discipline and mathematical physicists (or physical mathematicians, if I may say) have constantly crossed the boundaries between physics and mathematics. In its very essence, physics is mathematical and the work of Newton, Laplace, Fourier, Maxwell, Helmholtz and Poincaré, to name only a few, can be seen as belonging to mathematical physics. But it is only with the advent of Quantum Mechanics that the fusion of physics and mathematics has been attempted on a grand scale, with maybe the exception of General Relativity (in the work of H. Weyl, [8] especially). The physical theory of QM was borne by the efforts of such men as Born, Dirac, Jordan, Pauli, Schrödinger and Heisenberg. And although Schrödinger's wave formulation, Heisenberg's matrix formulation and Dirac's transformation theory are physico-mathematical constructions on their own right, my thesis is that Hilbert and von Neumann made a breakthrough in their work of 1926 and 1927 together with Nordheim "Ueber die Grundlagen der Quantenmechanik" [3] -- as a point of biographical interest, von Neumann and Nordheim were Hilbert's assistants at the time. Not only did they introduce the sharp separation between the mathematical formalism -- what Hilbert called "der analytische Apparat", the analytical apparatus -- and its physical interpretation, but they also gave a firm mathematical foundation to the concept of probability and it is this theme that I shall develop in the following.

2. THE AXIOMATIC METHOD IN PHYSICS

It is clear that the notion of analytical apparatus is drawn from the general structure of an axiomatic system and Hilbert makes no mystery of his intention to provide physics with the same kind of axiomatic foundations as geometry. Physical situations must be mirrored in an analytical apparatus, physical quantities are represented by mathematical constructs which are translated back into the language of physics in order to give real meaning to empirical statements. The analytical apparatus is not subjected to change while its physical interpretation has a variable degree of freedom or arbitrariness. What this means is that the mathematical formalism of a physical theory is a syntactical structure which does not possess a canonical interpretation, the analytical apparatus does not generate a unique model. At the same time, axiomatisation helps in clarifying a concept like probability which is thus rescued from its mystical state. It is noteworthy that another pair of renowned mathematicians, Hardy and Littlewood, expressed the same opinion at about the same time: "Probability is not a notion of pure mathematics, but of philosophy or physics" [2]. What then is so special about the notion of probability?

3. PROBABILITIES

Probabilities had, long before Quantum Mechanics, been knocking at the door of physics, but Laplace had entitled his work *Essai philosophique sur les probabilités* (1814) after having called it *Théorie analytique des probabilités* (1812). Statistical mechanics can certainly count as a forerunner of QM as far as the statistical behavior of a large number of particles is an essential ingredient in the probability theory of quantum-mechanical systems. But even in the work of pioneers like Born and Pauli, probability has entered QM somehow through the backdoor and it seems that it is only reluctantly that Born, for example, has admitted the idea of probability (cf. M. Jammer [4]). Later work by Kolmogorov on the axiomatic foundations of elementary probability theory or von Mises and Reichenbach [5] on the frequentist interpretation of probability will achieve some measure of success, but it is the historical event of a rigorous formalisation of the notion of probability as it occurs in quantum physics which has not been sufficiently stressed, in my opinion. In line with what I have said at the beginning, I would like now to draw the attention to the mathematical origin of quantum-mechanical probabilities.

4. MATHEMATICAL FOUNDATIONS

If probability has evidently a multiple application in QM, it remains that it is mainly a mathematical notion. Von Neumann's work in 1927-1932 (see [6] and [7]) focuses on what is called the finiteness of the eigenvalue problem. The point here is that any calculation is finite and since we have only finite results, these must be the products of finite calculation which is itself made possible only if the analytical apparatus contains the mathematical structures which enable such calculations. Such a formalism is the complex Hilbert space with

$$|\psi|^2 \qquad L^2(\mu)$$

where μ is a real positive measure on the functional space L^2 (i.e. the equivalence class of square-integrable functions). The integral

$$\int |\psi|^2 \, dv$$

is finite, which is equivalent to the fact that the sum

$$\sum_{n=1}^{\infty} |x_n|^2$$

of all sequences x_1, x_2, ... (of complex numbers) is finite in an orthonormal system of vectors. This mathematical fact, which Hilbert derived in the theory of integral equations in 1907, states that a linear expression

$$a_1x_1 + a_2x_2 + \ldots$$

is a linear function, if and only if the sum of the squares of the coefficients in the linear expression $a_1, a_2, \ldots$ is finite. The theorem, inspired by Kronecker's result on linear forms (homogeneous polynomials) is the very basis of the Hilbert space formulation of QM. Notice that on the probabilistic or statistical interpretation, the "acausal" interaction between an observed system and an observing system takes place in a given experimental situation and produces a univocal result of finite statistics for real or realised measurements.

In order for real measurements to have real positive probability values, the analytical apparatus must satisfy certain realizability conditions, <*Realitätsbedingungen*>, as Hilbert and von Neumann put it. For example, orthogonality for vectors, linearity and hermiticity for functional operators and the finiteness of the eigenvalue problem for Hermitean operators, as in von Neumann's further work *Mathematische Grundlagen der Quantenmechanik* are such constraints of realizability. I do not want to insist upon this, but I link all those constraints to a finitist or a constructivist point of view that ultimately reaches back to Kronecker's arithmetical constructivism in the second half of the 19th century, which required that all mathematics be founded on finite or effective arithmetic. The foundational import of Kronecker's influence can only be sketched here (see my [1]).

Although many people have contributed to the theory of functional analysis, among them Banach, Weyl, Wiener, the mathematician with the greatest influence on von Neumann was certainly Hilbert. Hilbert's foundational attitude is generally described as "finitism" <*finite Einstellung*>. As one of the foremost mathematicians of his time, Hilbert has expressed himself on the foundations of mathematics at various stages of his career. His philosophy of mathematics was supposed to be formalist at one time, but his mature views on the subject are summarized in a 1930 acknowledgement:

> Kronecker has clearly formulated a conception which he has made explicit in numerous examples: his conception corresponds essentially to our finitist viewpoint.

What are the finitist foundations of mathematics? In Kronecker's terminology, finistist foundations correspond to what he called <*allgemeine Arithmetik*> or general arithmetic. General arithmetic is the arithmetic of natural numbers with its algebraic extensions, that is the arithmetical theory of algebraic numbers. General arithmetic seems to exclude transcendental numbers or indeterminate numbers ("die Unbestimmte") as Kronecker would say, but Kronecker is more interested in the arithmetical core of transcendental quantities and he would try to extract some determinate relationship from transcendental functions, for example. Arithmetic is then the ultimate foundation on which the whole edifice of mathematics must rest. Hilbert conceives the finitary ideal of arithmetic in the same manner, but he also wants to make room for non-finitary means. Arithmetization of analysis is a desideratum for him as much as it is for Kronecker and Hilbert will even say that arithmetization of geometry is achieved in non-Euclidean geometry through the direct introduction of the number concept. Number means natural, rational and algebraic numbers and Kronecker had shown how to reduce the various concepts of number to the "natural" field, that is the field of natural numbers and its algebraic extensions. As we know, Hilbert will attempt a purely axiomatic justification of transfinite arithmetic with the help of ideal elements in order to rescue the paradise Cantor has created for us, as he says. But his metamathematics or theory of formal systems was meant to continue beyond arithmetic and into analysis by other means. Those means were logical. The intuitiveness of finite arithmetic was relayed by logical laws supposed to be as evident as arithmetic.

When Hilbert explains in his conference of 1925 "*Ueber das Unendliche*" "On the Infinite", that from a finitary point of view <*finiter Standpunkt*>, there are are two kinds of formulas in mathematics, the first ones corresponding to finitary statements and the second ones to ideal structures -- which are deprived of meaning, "sinnlos" -- he simply translates Kronecker's language of a pure arithmetic and its indeterminate extensions (with ideal elements) into the metamathematics or proof theory he hopes to build. Logic must insure the passage from finite arithmetic to transfinite arithmetic and analysis and Hilbert devised a logical (transfinite) choice function (named ε) to bridge the gap between the finite and the infinite. The problem of consistency called for the final legitimation of classical mathematics, but, as we know, that goal could not be achieved.

It seemed to me important to delineate Hilbert's stance in the foundations of mathematics, since it helps explain central features of his contribution to mathematical physics -- one should note that Hilbert has contributed to other topics in mathematical physics including his work co-authored with Richard Courant on *Mathematical Methods in Physics*. Undoubtedly, Hilbert's influence is noticeable in von Neumann's contributions to set theory and logic, but these remain scarce and von Neumann has not developed a position of his own on that matter, although he has contributed to many other subjects besides Quantum Mechanics, to mention only the theory of automata and quantum logic. Von Neumann's well-known proof of the impossibility of hidden variables -- as reworked with Birkhoff in 1936 -- can be seen as the birthday of the subject. Von Neumann came back to quantum logic at the end of his life and he discussed in his paper "Quantum Logics" the case of a continuous geometry without points and whose elements are all the linear subspaces of a given space; von Neumann thought that the logic of quantum probabilities (frequencies) could be built upon such a geometry. But here the probability measures must be infinite in order to be convergent and the probability statements that express those measures are required to have a finite meaning, as Reichenbach claimed for the verifiability theory of his probability theory [5].

To sum up, Hilbert and von Neumann can be said to have crossed many boundaries in mathematics, physics, foundations of mathematics, logic and even philosophy. Their pioneering work in the foundations of Quantum Mechanics is but one example of the intellectual proficiency of both mathematicians. Quantum Mechanics was not the major field of activity for Hilbert, but it certainly was the arena for von Neumann's most important contributions to science. Hilbert provided the mathematical and philosophical background while von Neumann embarked in the business of offering a full account of the analytical apparatus of QM. Whether Hilbert and von Neumann have succeeded in putting QM on a firm basis, it is to the physicist and the philosopher of physics to decide, since the latter is the one in charge of the validation of theories with respect to their applicability, efficiency and relevance for contemporary science and philosophy.

5. CONCLUSION

From a historical point of view, the boundaries crossed by Hilbert and von Neumann might have an import only for those interested in the history of physics or mathematics. But their historical alliance has a broader intellectual significance, if one allows for philosophical perspective. For the philosopher of science, the joint paper of 1927 and the subsequent work of von Neumann mark a new era in the foundations of physics and although it has been heralded by only a few historians or philosophers of science, mathematicians and physicists have put it at the very center of their work. Von Neumann's elaboration of the Hilbert space formalism for Quantum Mechanics is still the conceptual foundation of quantum physics and although it has been criticized by some realist philosophers or physicists-philosophers, (another cross-bred species, besides mathematical physicists, physical mathematicians, philosophical mathematicians and mathematical philosophers, etc.) as an instrumentalist contraption, there is no doubt that the rigorous foundations of QM must be attributed to von Neumann and Hilbert. The main philosophical lesson to be drawn in that context is the idea that an analytical apparatus, a formalism, is not canonical or does not generate its own interpretation. Realism, litteral or critical, should learn from mathematical physics, especially the historical episode I have evoked, that bridges between mathematics and physics can take many shapes or that physical reality can be modeled in more ways than one. The constructivist or anti-realist philosopher of science, who has weak ties with the constructivist school in the sociology of knowledge, a school that originated, as I understand, here in Edinburgh, the constructivist philosopher or historian of science pretends that it is the workings of finite agents equipped with their finite conceptual or linguistic structures which shape a world, physical, mathematical and real, which is otherwise undetermined. I have offered only an example in the history of physics, an example that is also an episode in the cultural history of the XXth century. I hope to have shown, if only very briefly, that Hilbert and von Neumann are examplary as individuals crossing boundaries.

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