On the Breitung Test for Panel Unit Roots and Local Asymptotic Power: Furthr Calculations

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This unpublished appendix contains details of some algebraic results that were skipped over in the paper.

1 Calculation of $Var\left[\frac{1}{T}\sum_{t=2}^{T-1} s_t\left\{\varepsilon_{it}-\frac{1}{T-t}\left(x_{iT}^*-x_{it}^*\right)\right\}\left\{x_{it-1}^*-\frac{t-1}{T}x_{iT}^*\right\}\right]$:

Using the definition of x_{it}^*, the quantity of interest is:

$$
\begin{aligned}
& E\left[\frac{1}{T}\sum_{t=2}^{T-1} s_t\left\{\varepsilon_{it}-\frac{1}{T-t}\left(\sum_{r=t+1}^{T}\varepsilon_{ir}\right)\right\}\left\{\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)-\frac{t-1}{T}\left(\sum_{s=1}^{T}\varepsilon_{is}\right)\right\}\right]^2 \\
= & E\left[\begin{array}{c}\frac{1}{T}\sum_{t=2}^{T-1} s_t\varepsilon_{it}\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)-\frac{1}{T}\sum_{t=2}^{T-1} s_t\frac{1}{T-t}\left(\sum_{r=t+1}^{T}\varepsilon_{ir}\right)\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right) \\ -\frac{1}{T}\sum_{t=2}^{T-1} s_t\varepsilon_{it}\frac{t-1}{T}\left(\sum_{s=1}^{T}\varepsilon_{is}\right)+\frac{1}{T}\sum_{t=2}^{T-1} s_t\frac{1}{T-t}\frac{t-1}{T}\left(\sum_{r=t+1}^{T}\varepsilon_{ir}\right)\left(\sum_{s=1}^{T}\varepsilon_{is}\right)\end{array}\right]^2.
\end{aligned}
$$

There are 10 terms to calculate:

- (1)

$$
\begin{aligned}
E\left[\frac{1}{T}\sum_{t=2}^{T-1} s_t\varepsilon_{it}\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)\right]^2 &= \frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\left(\sum_{s=1}^{t-1}\sum_{q=1}^{p-1}\right)s_p s_t E\left[\varepsilon_{it}\varepsilon_{is}\varepsilon_{ip}\varepsilon_{iq}\right] \\
&= \frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{s=1}^{t-1} s_t^2+O\left(\frac{1}{T}\right)=\frac{1}{T}\sum_{t=2}^{T-1}\frac{t-1}{T}s_t^2+O\left(\frac{1}{T}\right) \\
&= \int_0^1 r dr=\frac{1}{2}.
\end{aligned}
$$

- (2)

$$
\begin{aligned}
& E\left[\frac{1}{T^2}\sum_{t=2}^{T-1} s_t \frac{1}{1-\frac{t}{T}}\left(\sum_{r=t+1}^{T}\varepsilon_{ir}\right)\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)\right]^2 \\
= & \frac{1}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1} s_t \frac{1}{1-\frac{t}{T}} s_p \frac{1}{1-\frac{p}{T}} \sum_{r=t+1}^{T}\sum_{q=p+1}^{T}\sum_{s=1}^{t-1}\sum_{w=1}^{p-1} E\left[\varepsilon_{ir}\varepsilon_{iq}\varepsilon_{is}\varepsilon_{iw}\right] \\
= & \frac{1}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1} s_t \frac{1}{1-\frac{t}{T}} s_p \frac{1}{1-\frac{p}{T}} \sum_{r=\max(t,p)+1}^{T}\ \sum_{s=1}^{\min(t,p)-1} +O\left(\frac{1}{T}\right) \\
= & \frac{1}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1} s_t \frac{1}{1-\frac{t}{T}} s_p \frac{1}{1-\frac{p}{T}} \left(T-\max\left(t,p\right)\right)\left(\min\left(t,p\right)-1\right)+O\left(\frac{1}{T}\right) \\
\to & \int_0^1\int_0^1 \frac{1}{1-r}\frac{1}{1-s}\left(1-\max\left(r,s\right)\right)\min\left(r,s\right)dsdr \\
= & \int_0^1\int_0^r \frac{1}{1-r}\frac{1}{1-s}\left(1-r\right)sdsdr+\int_0^1\int_r^1 \frac{1}{1-r}\frac{1}{1-s}\left(1-s\right)rdsdr \\
= & \int_0^1\int_0^r \frac{s}{1-s}\,ds\,dr+\int_0^1\int_r^1 \frac{r}{1-r}dsdr \\
= & \frac{1}{2}+\frac{1}{2}=1.
\end{aligned}
$$

- (3)

$$
\begin{aligned}
& E\left[\frac{1}{T}\sum_{t=2}^{T-1} s_t\varepsilon_{it}\frac{t-1}{T}\left(\sum_{s=1}^{T}\varepsilon_{is}\right)\right]^2 \\
= & \frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{s=1}^{T}\sum_{q=1}^{T} s_t s_p \frac{t-1}{T}\frac{p-1}{T} E\left[\varepsilon_{it}\varepsilon_{ip}\varepsilon_{is}\varepsilon_{iq}\right] \\
= & \frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{s=1}^{T} s_t^2\left(\frac{t-1}{T}\right)^2+\frac{2}{T^2}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1} s_t s_p \frac{t-1}{T}\frac{p-1}{T}+O\left(\frac{1}{T}\right). \\
\to & \int_0^1 r^2dr+2\int_0^1\int_0^1 rpdpdr \\
= & \frac{5}{6}.
\end{aligned}
$$

- (4)

$$
\begin{aligned}
& E\left[\frac{1}{T}\sum_{t=2}^{T-1}s_t\frac{1}{T-t}\frac{t-1}{T}\left(\sum_{r=t+1}^{T}\varepsilon_{ir}\right)\left(\sum_{s=1}^{T}\varepsilon_{is}\right)\right]^2 \\
= \quad & \frac{1}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{s=1}^{T}\sum_{w=1}^{T}\sum_{r=t+1}^{T}\sum_{q=p+1}^{T}s_t\frac{1}{1-\frac{t}{T}}\frac{t-1}{T}s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}E\left[\varepsilon_{ir}\varepsilon_{is}\varepsilon_{iq}\varepsilon_{iw}\right] \\
= \quad & \frac{2}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{s=t+1}^{T}\sum_{w=p+1}^{T}s_t\frac{1}{1-\frac{t}{T}}\frac{t-1}{T}s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}+O\left(\frac{1}{T}\right) \\
& +\frac{1}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{s=1}^{T}\sum_{w=\max(t,p)+1}^{T}s_t\frac{1}{1-\frac{t}{T}}\frac{t-1}{T}s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}+O\left(\frac{1}{T}\right) \\
\rightarrow \quad & 2\int_0^1\int_0^1\int_r^1\int_p^1\frac{r}{1-r}\frac{p}{1-p}dwdsdpdr+\int_0^1\int_0^1\int_0^1\int_{\max(r,p)}^1\frac{r}{1-r}\frac{p}{1-p}dwdsdpdr \\
= \quad & \frac{1}{2}+\int_0^1\int_0^1\int_{\max(r,p)}^1\frac{r}{1-r}\frac{p}{1-p}dsdpdr \\
= \quad & \frac{1}{2}+\int_0^1\int_0^r\int_r^1\frac{r}{1-r}\frac{p}{1-p}dsdpdr+\int_0^1\int_r^1\int_p^1\frac{r}{1-r}\frac{p}{1-p}dsdpdr \\
= \quad & \frac{1}{2}+\int_0^1\int_0^r\frac{rp}{1-p}dpdr+\int_0^1\int_r^1\frac{rp}{1-r}dpdr \\
= \quad & \frac{1}{2}+\frac{5}{12}+\frac{5}{12}=\frac{4}{3}.
\end{aligned}
$$

- (5)

$$
\begin{aligned}
& -E\left[\frac{1}{T}\sum_{t=2}^{T-1}s_t\varepsilon_{it}\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)\right]\left[\frac{1}{T^2}\sum_{p=2}^{T-1}s_p\frac{1}{1-\frac{p}{T}}\left(\sum_{q=p+1}^{T}\varepsilon_{iq}\right)\left(\sum_{w=1}^{p-1}\varepsilon_{iw}\right)\right] \\
= \quad & -\frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{q=p+1}^{T}\sum_{s=1}^{t-1}\sum_{w=1}^{p-1}s_ts_p\frac{1}{1-\frac{p}{T}}E\left(\varepsilon_{it}\varepsilon_{iq}\varepsilon_{is}\varepsilon_{iw}\right) \\
= \quad & -\frac{1}{T^3}\sum_{p=2}^{T-1}\sum_{t=p+1}^{T}\sum_{s=1}^{p-1}s_ts_p\frac{1}{1-\frac{p}{T}}\ (t=q)(s=w)+O\left(\frac{1}{T}\right) \\
\rightarrow \quad & -\int_0^1\int_p^1\int_0^p\frac{1}{1-p}dsdrdp \\
= \quad & -\frac{1}{2}.
\end{aligned}
$$

- (6)

$$\begin{aligned}
&-E\left[\frac{1}{T}\sum_{t=2}^{T-1}s_t\varepsilon_{it}\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)\right]\left[\frac{1}{T}\sum_{p=2}^{T-1}s_p\varepsilon_{ip}\frac{p-1}{T}\left(\sum_{q=1}^{T}\varepsilon_{iq}\right)\right] \\
= \quad & -\frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}s_t s_p\frac{p-1}{T}\sum_{s=1}^{t-1}\sum_{q=1}^{T}E\left(\varepsilon_{it}\varepsilon_{is}\varepsilon_{ip}\varepsilon_{iq}\right) \\
= \quad & -\frac{1}{T^2}\sum_{t=2}^{T-1}s_t^2\frac{t-1}{T}\sum_{s=1}^{t-1}\ \text{(t=p) (s=q)}\ \left(=-\frac{1}{T}\sum_{t=2}^{T-1}s_t^2\left(\frac{t-1}{T}\right)^2\right) \\
& -\frac{1}{T^2}\sum_{t=3}^{T-1}\sum_{p=2}^{t-1}s_t s_p\frac{p-1}{T}+O\left(\frac{1}{T}\right)\ \text{(t=q) (s=p)} \\
\rightarrow \quad & -\int_0^1 r^2dr-\int_0^1\int_0^r pdpdr \\
= \quad & -\frac{1}{2}.
\end{aligned}$$

- (7)

$$\begin{aligned}
& E\left[\frac{1}{T}\sum_{t=2}^{T-1}s_t\varepsilon_{it}\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)\right]\left[\frac{1}{T^2}\sum_{p=2}^{T-1}s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}\left(\sum_{q=p+1}^{T}\varepsilon_{iq}\right)\left(\sum_{w=1}^{T}\varepsilon_{iw}\right)\right] \\
= \quad & \frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{s=1}^{t-1}\sum_{q=p+1}^{T}\sum_{w=1}^{T}s_t s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}E\left(\varepsilon_{it}\varepsilon_{is}\varepsilon_{iq}\varepsilon_{iw}\right) \\
= \quad & \frac{1}{T^2}\sum_{p=2}^{T-1}\sum_{t=p+1}^{T-1}s_t s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}\frac{t-1}{T}\ \text{(t=q) (s=w)} \\
& +\frac{1}{T^3}\sum_{t=4}^{T-1}\sum_{s=3}^{t-1}\sum_{p=2}^{s-1}s_t s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}+O\left(\frac{1}{T}\right)\ \text{(t=w) (s=q)} \\
\rightarrow \quad & \int_0^1\int_p^1\frac{pr}{1-p}drdp+\int_0^1\int_0^r\int_0^s\frac{p}{1-p}dpdsdr \\
= \quad & \frac{5}{12}+\int_0^1\int_0^r\int_0^s\frac{p}{1-p}\,dp\,ds\,dr \\
= \quad & \frac{5}{12}+\frac{1}{12}=\frac{1}{2}.
\end{aligned}$$

- (8)

$$
\begin{aligned}
& E\left[\frac{1}{T^2}\sum_{t=2}^{T-1}s_t\frac{1}{1-\frac{t}{T}}\left(\sum_{r=t+1}^{T}\varepsilon_{ir}\right)\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)\right]\left[\frac{1}{T}\sum_{p=2}^{T-1}s_p\varepsilon_{ip}\frac{p-1}{T}\left(\sum_{q=1}^{T}\varepsilon_{iq}\right)\right] \\
= & \frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{r=t+1}^{T}\sum_{s=1}^{t-1}\sum_{q=1}^{T}s_ts_p\frac{1}{1-\frac{t}{T}}\frac{p-1}{T}E\left(\varepsilon_{ir}\varepsilon_{is}\varepsilon_{ip}\varepsilon_{iq}\right) \\
= & \frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{p=t+1}^{T-2}\sum_{s=1}^{t-1}s_ts_p\frac{1}{1-\frac{t}{T}}\frac{p-1}{T}\ \text{(r=p) (s=q)} \\
& +\frac{1}{T^2}\sum_{t=3}^{T-1}\sum_{p=2}^{t-1}s_ts_p\frac{1}{1-\frac{t}{T}}\frac{p-1}{T}\frac{T-t}{T}\ \text{(r=q) (s=p)} \\
= & \int_0^1\int_r^1\int_0^r\frac{p}{1-r}dsdpdr+\int_0^1\int_0^r pdpdr \\
= & \frac{7}{12}.
\end{aligned}
$$

- (9)

$$
\begin{aligned}
& -E\left[\frac{1}{T^2}\sum_{t=2}^{T-1}s_t\frac{1}{1-\frac{t}{T}}\left(\sum_{r=t+1}^{T}\varepsilon_{ir}\right)\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right)\right]\left[\frac{1}{T^2}\sum_{p=2}^{T-1}s_p\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}\left(\sum_{q=p+1}^{T}\varepsilon_{iq}\right)\left(\sum_{w=1}^{T}\varepsilon_{iw}\right)\right] \\
= & -\frac{1}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{r=t+1}^{T}\sum_{s=1}^{t-1}\sum_{q=p+1}^{T}\sum_{w=1}^{T}s_ts_p\frac{1}{1-\frac{t}{T}}\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}E\left(\varepsilon_{ir}\varepsilon_{is}\varepsilon_{iq}\varepsilon_{iw}\right) \\
= & -\frac{1}{T^4}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{q=\max(t,p)+1}^{T}\sum_{s=1}^{t-1}s_ts_p\frac{1}{1-\frac{t}{T}}\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}\ \text{(r=q) (s=w)} \\
& -\frac{1}{T^3}\sum_{p=2}^{T-3}\sum_{s=p+1}^{T-2}\sum_{t=s+1}^{T-1}s_ts_p\frac{1}{1-\frac{t}{T}}\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}\left(1-\frac{t}{T}\right)+O\left(\frac{1}{T}\right)\ \text{(r=w) (s=q)} \\
\to & -\int_0^1\int_0^1\int_{\max(r,p)}^1\int_0^r\frac{1}{1-r}\frac{p}{1-p}dsdqdpdr-\int_0^1\int_p^1\int_s^1\frac{1}{1-r}\frac{1}{1-p}p\left(1-r\right)drdsdp \\
= & -\int_0^1\int_0^1\int_{\max(r,p)}^1\frac{r}{1-r}\frac{p}{1-p}dsdpdr-\int_0^1\int_p^1\int_s^1\frac{1}{1-r}\frac{1}{1-p}p\left(1-r\right)drdsdp \\
= & -\int_0^1\int_0^1\int_{\max(r,p)}^1\frac{r}{1-r}\frac{p}{1-p}dsdpdr-\frac{1}{12} \\
= & -\frac{5}{6}-\frac{1}{12}=-\frac{11}{12}.
\end{aligned}
$$

- (10)

$$
\begin{aligned}
& -E\left[\frac{1}{T}\sum_{t=2}^{T-1} s_t \varepsilon_{it}\frac{t-1}{T}\left(\sum_{s=1}^{T}\varepsilon_{is}\right)\right]\left[\frac{1}{T^2}\sum_{p=2}^{T-1} s_p \frac{1}{1-\frac{p}{T}}\frac{p-1}{T}\left(\sum_{q=p+1}^{T}\varepsilon_{iq}\right)\left(\sum_{w=1}^{T}\varepsilon_{iw}\right)\right] \\
= & -\frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1}\sum_{s=1}^{T}\sum_{q=p+1}^{T}\sum_{w=1}^{T} s_t s_p \frac{t-1}{T}\frac{1}{1-\frac{p}{T}}\frac{p-1}{T} E\left(\varepsilon_{it}\varepsilon_{is}\varepsilon_{iq}\varepsilon_{iw}\right) \\
= & -\frac{2}{T^2}\sum_{t=2}^{T-1}\sum_{p=2}^{T-1} s_t s_p \frac{t-1}{T}\frac{1}{1-\frac{p}{T}}\frac{p-1}{T}\frac{T-p}{T} \text{ (t=s) (q=w), (t=w) (s=q)} \\
& -\frac{1}{T^2}\sum_{p=2}^{T-2}\sum_{t=p+1}^{T-1} s_t s_p \frac{t-1}{T}\frac{1}{1-\frac{p}{T}}\frac{p-1}{T} \text{ (t=q) (s=w)} \\
\to & -2\int_0^1\int_0^1 r\frac{1}{1-p}p\left(1-p\right)dpdr - \int_0^1\int_p^1 r\frac{p}{1-p}drdp \\
= & -\frac{11}{12}.
\end{aligned}
$$

- Summing up, we have

$$
\begin{aligned}
& \frac{1}{2}+1+\frac{5}{6}+\frac{4}{3}+2\left(-\frac{1}{2}-\frac{1}{2}+\frac{1}{2}+\frac{7}{12}-\frac{11}{12}-\frac{11}{12}\right) \\
= & \frac{6+12+10+16-12+14-22-22}{12} \\
= & \frac{1}{6}.\blacksquare
\end{aligned}
$$

2 Decomposition of B_{it-1} and D_{it-1} :

For $t = 2, ..., T-1$, we can write

$$
\begin{aligned}
B_{it} &= s_t\left[x_{it-1} - \frac{1}{T-t}\left(\sum_{q=t}^{T-1} x_{iq}\right)\right] \\
&\simeq s_t\left[x^*_{it-1} - \frac{1}{T-t}\left(\sum_{q=t}^{T-1} x^*_{iq}\right)\right] \\
&\quad -\frac{c_i}{n^{1/4}}s_t\left[\sum_{p=0}^{t-2}\left(\frac{t-p-1}{T}\right)\varepsilon_{ip} - \frac{1}{T-t}\left(\sum_{q=t}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)\varepsilon_{ip}\right)\right] + o\left(1\right) \\
&= B_{1,it} - \frac{c_i}{n^{1/4}}B_{2,it}, \text{ say,}
\end{aligned}
$$

$$
\begin{aligned}
D_{it-1} &= \sum_{q=0}^{t-2} x_{iq} - \frac{t-1}{T}\sum_{q=0}^{T-1} x_{iq} \\
&\simeq \sum_{q=0}^{t-2} x_{iq}^{*} - \frac{t-1}{T}\sum_{q=0}^{T-1} x_{iq}^{*} \\
&\quad -\frac{c_i}{n^{1/4}}\left[\sum_{q=1}^{t-2}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)\varepsilon_{ip} - \frac{t-1}{T}\sum_{q=1}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)\varepsilon_{ip}\right] \\
&= D_{1,it-1} - \frac{c_i}{n^{1/4}} D_{2,it-1}, \text{ say.}
\end{aligned}
$$

3 Calculation of $E\left[\frac{1}{T^2}\sum_{t=2}^{T-1} A_{it}D_{2,it-1}\right]$:

$$
\begin{aligned}
& E\left[\frac{1}{T^2}\sum_{t=2}^{T-1} A_{it}D_{2,it-1}\right] \\
= \;& E\frac{1}{T^2}\sum_{t=2}^{T-1}\left[s_t\left(\varepsilon_{it} - \frac{1}{T-t}\left(\sum_{q=t+1}^{T}\varepsilon_{iq}\right)\right)\right]\left[\sum_{q=1}^{t-2}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)\varepsilon_{ip} - \frac{t-1}{T}\sum_{q=1}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)\varepsilon_{ip}\right] \\
= \;& -\frac{1}{T^2}\sum_{t=2}^{T-1} s_t\frac{t-1}{T}E\left[\left(\varepsilon_{it} - \frac{1}{T-t}\left(\sum_{q=t+1}^{T}\varepsilon_{iq}\right)\right)\left(\sum_{q=1}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)\varepsilon_{ip}\right)\right] \\
= \;& -\frac{1}{T^2}\sum_{t=2}^{T-1} s_t\frac{t-1}{T}\sum_{q=1}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)E\left(\varepsilon_{it}\varepsilon_{ip}\right) + \frac{1}{T^2}\sum_{t=2}^{T-1} s_t\frac{t-1}{T}\frac{1}{T-t}\sum_{s=t+1}^{T}\sum_{q=1}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)E\left(\varepsilon_{is}\varepsilon_{ip}\right) \\
= \;& -\frac{1}{T^2}\sum_{t=2}^{T-1} s_t\frac{t-1}{T}\sum_{p=0}^{t-1}\left(\frac{t-p}{T}\right) + \frac{1}{T^2}\sum_{t=2}^{T-3}\sum_{p=t+1}^{T-2}\sum_{q=p+1}^{T-1} s_t\frac{t-1}{T}\frac{1}{T-t}\left(\frac{q-p}{T}\right) \\
\to \;& -\int_0^1\int_0^r r\left(r-s\right)dsdr + \int_0^1\int_r^1\int_p^1 \frac{r}{1-r}\left(q-p\right)dqdpdr \\
= \;& -\frac{1}{36}.
\end{aligned}
$$

4 Calculation of $E\left[\frac{1}{T^2}\sum_{t=2}^{T-1} B_{2,it}C_{it-1}\right]$:

$$
\begin{aligned}
& E\left[\frac{1}{T^2}\sum_{t=2}^{T-1} B_{2,it}C_{it-1}\right] \\
= \; & E\frac{1}{T^2}\sum_{t=2}^{T-1}\left[s_t\left[\sum_{p=0}^{t-2}\left(\frac{t-p-1}{T}\right)\varepsilon_{ip} - \frac{1}{T-t}\left(\sum_{q=t}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)\varepsilon_{ip}\right)\right]\right] \\
& \times\left[\left(\sum_{s=1}^{t-1}\varepsilon_{is}\right) - \frac{t-1}{T}\left(\sum_{s=1}^{T}\varepsilon_{is}\right)\right] \\
= \; & \frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{s=1}^{t-1}\left[s_t\sum_{p=0}^{t-2}\left(\frac{t-p-1}{T}\right)E\left(\varepsilon_{ip}\varepsilon_{is}\right)\right] - \frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{s=1}^{t-1}\left[s_t\frac{1}{T-t}\sum_{\substack{q=t \\ (s=p<t<q)}}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)E\left(\varepsilon_{ip}\varepsilon_{is}\right)\right] \\
& -\frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{s=1}^{T}\left[s_t\sum_{p=0}^{t-2}\left(\frac{t-p-1}{T}\right)\frac{t-1}{T}E\left(\varepsilon_{ip}\varepsilon_{is}\right)\right] \\
& +\frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{s=1}^{T}\left[s_t\frac{1}{T-t}\frac{t-1}{T}\sum_{\substack{q=t \\ ((p=s),t<q)}}^{T-1}\sum_{p=0}^{q-1}\left(\frac{q-p}{T}\right)E\left(\varepsilon_{ip}\varepsilon_{is}\right)\right] \\
= \; & \frac{1}{T^2}\sum_{t=2}^{T-1}\left[s_t\sum_{p=1}^{t-2}\frac{t-p-1}{T}\right] - \frac{1}{T^2}\sum_{p=1}^{T-3}\sum_{t=p+1}^{T-2}\sum_{q=t+1}^{T-1}s_t\frac{1}{T-t}\frac{q-p}{T} \\
& -\frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{s=1}^{t-2}\left[s_t\left(\frac{t-s-1}{T}\right)\frac{t-1}{T}\right] + \frac{1}{T^2}\sum_{t=2}^{T-1}\sum_{p=0}^{T-2}\sum_{q=\max\{t,p\}}^{T-1}s_t\frac{1}{T-t}\frac{t-1}{T}\frac{q-p}{T} \\
\to \; & \int_0^1\int_0^r (r-s)\,dsdr - \int_0^1\int_p^1\int_r^1 \frac{1}{1-r}(q-p)\,dqdrdp - \int_0^1\int_0^r (r-s)\,rdsdr \\
& + \int_0^1\int_0^1\int_{\max\{r,p\}}^1 \frac{r}{1-r}(q-p)\,dqdpdr \\
= \; & -\frac{1}{36}
\end{aligned}
$$

5 Calculation of $E\left[\frac{1}{T^3}\sum_{t=2}^{T-1}E\left(B_{1,it}D_{1,it-1}\right)\right]$:

$$
\begin{aligned}
& E\left[\frac{1}{T^3}\sum_{t=2}^{T-1}E\left(B_{1,it}D_{1,it-1}\right)\right] \\
= \; & E\frac{1}{T^3}\sum_{t=2}^{T-1}s_t\left[x_{it-1}^{*}-\frac{1}{T-t}\left(\sum_{q=t}^{T-1}x_{iq}^{*}\right)\right]\left[\sum_{s=0}^{t-2}x_{is}^{*}-\frac{t-1}{T}\sum_{s=0}^{T-1}x_{is}^{*}\right] \\
= \; & \frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{s=0}^{t-2}s_tE\left(x_{it-1}^{*}x_{is}^{*}\right)-\frac{1}{T^3}\sum_{t=2}^{T-1}s_t\frac{1}{T-t}\sum_{q=t}^{T-1}\sum_{s=0}^{t-2}E\left(x_{iq}^{*}x_{is}^{*}\right) \\
& -\frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{s=0}^{T-1}s_t\frac{t-1}{T}E\left(x_{it-1}^{*}x_{is}^{*}\right)+\frac{1}{T^3}\sum_{t=2}^{T-1}s_t\frac{1}{T-t}\frac{t-1}{T}\sum_{q=t}^{T-1}\sum_{s=0}^{T-1}E\left(x_{iq}^{*}x_{is}^{*}\right) \\
= \; & \frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{s=0}^{t-2}s_t\min\left(t-1,s\right)-\frac{1}{T^3}\sum_{t=2}^{T-1}s_t\frac{1}{T-t}\sum_{q=t}^{T-1}\sum_{s=0}^{t-2}\min\left(q,s\right) \\
& -\frac{1}{T^3}\sum_{t=2}^{T-1}\sum_{s=0}^{T-1}s_t\frac{t-1}{T}\min\left(t-1,s\right)+\frac{1}{T^3}\sum_{t=2}^{T-1}s_t\frac{1}{T-t}\frac{t-1}{T}\sum_{q=t}^{T-1}\sum_{s=0}^{T-1}\min\left(q,s\right)+O\left(\frac{1}{T}\right) \\
\rightarrow \; & \int_0^1\int_0^r sdsdr-\int_0^1\int_r^1\int_0^r\frac{s}{1-r}dsdqdr-\int_0^1\int_0^1 r\min\left(r,s\right)dsdr \\
& +\int_0^1\int_0^1\int_r^1\frac{r}{1-r}\min\left(s,q\right)dqdsdr \\
= \; & \frac{1}{36}.
\end{aligned}
$$