Asymptotic Local Power of pooled $t-$ ratio Tests for Unit Roots in Panels with Fixed Effects*

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Abstract

We derive analytically the local asymptotic power of two pooled t-ratio tests for the presence of a unit root in a panel with fixed effects. We consider two statistics which differ according to the method used to remove the bias of the pooled OLS estimator. We show that when we bias-correct the numerator only, the resulting test has significant local power in $n^{-1/4}T^{-1}$ neighborhoods of the null of a panel unit root, while when the entire estimator is corrected for bias, the resulting statistic has local asymptotic power in neighborhoods shrinking at the faster rate of $n^{-1/2}T^{-1}$. This latter test is equivalent to the well-known pooled t test proposed by Levin, Lin, and Chu (2002), and its power depends only on the mean of the local-to-unity parameters. This implies that it has the same power against homogeneous and heterogeneous alternatives with the same mean autoregressive parameter. We then compare these tests to a panel version of the Sargan-Bhargava (1983) statistic for a unit root and the common point-optimal test of Moon, Perron, and Phillips (2007). Monte Carlo simulations confirm the usefulness of our local-to-unity framework.

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1 Introduction

Testing for a unit root in panels with large cross-sectional and time dimensions has been heavily studied in the past decade. Early contributions in this research area include Quah (1994) and Levin, Lin, and Chu (2002) who propose $t-$ ratio type tests, Im, Pesaran, and Shin (2003) who propose average type tests, and Maddala and Wu (1999) and Choi (2001) who propose Fisher-type statistics. These papers, however, derive the limits of the tests only under the null hypothesis of a unit root in all units in the panel. The power of these tests is studied through Monte Carlo simulations but not analytically.

The main purpose of this paper is to derive analytically the local asymptotic power of tests that are based on two different bias-adjusted pooled ordinary least squares (OLS) estimators. We concentrate on tests that are based on pooled OLS estimators for three reasons. First, this approach includes the well-known panel unit root tests proposed by Levin, Lin, and Chu (2002) (LLC thereafter) and Quah (1994). Secondly, and most importantly, there is no analytical result available on the power of the LLC test. This paper provides such an analytical expression, and this may enable more accurate statistical comparisons with other panel unit root tests. Finally, it is known that without fixed effects, the LLC test is uniformly most powerful (UMP) against a homogeneous stationary alternative (see Bowman (2002) and Moon, Perron, and Phillips (2007)). From this, several interesting theoretical questions arise such as whether it is still UMP with fixed effects and the impact of heterogeneity. If the LLC test is no longer UMP, we would like to quantify its power loss relative to both an optimal test and other available tests.

We use a local-to-unity framework in which the autoregressive parameters are different for each cross-section. The local-to-unity parameters are assumed to be drawn from some unspecified distribution. We express our limiting results in terms of the moments of this distribution. We also assume that the errors are independent in both time and cross-sectional dimensions. We consider two statistics based on a bias-adjusted pooled ordinary least squares (OLS) estimator. The first statistic we analyze (called $t^{\#}$ in the paper) is the t - ratio statistic studied in Section 3 of Moon and Perron (2004) and the second one (called t^{+} test in the paper) is equivalent to the statistic proposed by Levin, Lin, and Chu (2002) for a unit root in a panel with fixed effects. It is known that the pooled OLS estimator used in $t^{\#}$ is more efficient than the one in t^{+} (*e.g.*, Hahn and Kuersteiner, 2002).

The main contributions of the paper are as follows.

Firstly, we show that the appropriate local alternative of the $t^{\#}$ test is of order $n^{-1/4}T^{-1}$ around the null hypothesis of a panel unit root (where n and T denote the size of the cross-section and time dimensions, respectively). On the other hand, the t^{+} test (or LLC test) is a weighted average of the $t^{\#}$ statistic and a panel version of Sargan and Bhargava statistic (denoted by V_o in the paper). We show that the asymptotic local power of the test t^{+} comes from V_o, and that it has significant asymptotic local power in $n^{-1/2}T^{-1}$ neighborhoods of the null of a panel unit root. Note that this rate is the combination of the usual $n^{-1/2}$ cross-sectional rate and T^{-1} rate used for univariate nonstationary time series. Obviously these findings imply that the $t^{\#}$ statistic should not be used in practice in testing for a panel unit root. These results also illustrate an exceptional case where a t-test based on a more efficient estimator might have lower power than a t-test based on a less efficient estimator.

Secondly, we find that the asymptotic local power of t^{+} depends only on the first moment of the local-to-unity parameters and not on the heterogeneity of the local alternative. However, since Moon, Perron, and Phillips (2007) find that the power envelope is positively related to the degree of heterogeneity of the alternative hypothesis, this implies that the loss in power of the LLC test relative to the power envelope increases as the alternative hypothesis becomes more heterogeneous.

Finally, we compare the power of the t^{+} test with that of some existing tests: a panel version of the Sargan and Bhargava (1983) statistic, the common point-optimal test of Moon, Perron, and Phillips (2007), and the power envelope of the optimal invariant test. We show that, under our assumptions, there is a clear ranking of the various tests in terms of power with the Moon, Perron, and Phillips test being the most powerful.

There are other recent papers that derive the power of some panel unit root tests analytically. Moon and Perron (2004) find that in panels without fixed effects, the t - ratio statistic for a unit root in the panel has significant asymptotic local power in neighborhoods of the unit root null shrinking at a rate of $n^{-1/2}T^{-1}$. However, when fixed effects exist in the panel, they found that the t - ratio statistic that they proposed (which is our $t^{\#}$ statistic) does not have local power in a $n^{-\kappa}T^{-1}$- neighborhood of unity with $\kappa > 1/4$. Moon, Perron, and Phillips (2006) derive the local power properties of the unibiased test proposed by Breitung (2000). To test for unit roots in panels with incidental trends, *i.e.* deterministic trends with individually heterogeneous coefficients, Ploberger and Phillips (2002) propose an optimal invariant test that maximizes a weighted average of power and show that it has power within $n^{-1/4}T^{-1}$ neighborhoods of the null of a panel unit root. Moon, Perron, and Phillips (2007) derive the power envelope of panel unit root tests and compare the powers of various invariant panel unit root tests. On the other hand, Bowman (2002) studies the exact power of panel unit root tests against fixed alternative hypotheses. He characterizes the class of admissible tests for unit roots in panels and shows that the average-type tests of Im, Pesaran, and Shin (2003) and the test based on Fisher-type statistics in Maddala and Wu (1999) and Choi (2001) are not admissible. He also proposes a test statistic that is derived using arguments similar to those of Ploberger and Phillips (2002).

In simulations, we compare the power of the tests we consider. We illustrate that, as suggested by theory, there is a clear ranking of these tests in terms of power with the $t^{\#}$ test the least powerful and the common point-optimal test the most powerful against both homogeneous and heterogeneous alternatives. Moreover, the power achieved by each test is close to the power predicted by our theory.

The paper is organized as follows. In Section 2 we introduce a simple panel unit root model with fixed effects and the hypotheses of interest. Section 3 contains the definition of the statistics we analyze. Section 4 contains the main results on the local asymptotic power of the $t^{\#}$ and t^{+} tests, while section 5 provides some simulation evidence. Section 6 concludes the paper and the appendix contains all the technical proofs and derivations.

2 Model and Hypotheses to Test

Consider the following dynamic panel model with fixed effects:

$$\begin{aligned} z_{it} &= \alpha_i + y_{it};\ i = 1, ..., n \text{ and } t = 1, ..., T+1, \\ y_{it} &= \rho_i y_{it-1} + e_{it}, \end{aligned} \tag{1}$$

where $e_{it} \sim iid\ (0, \sigma^2)$ across i and over t with finite fourth moments. The coefficients α_i in (1) are fixed effect parameters that measure individual effects in the panel. The dynamic panel model (1) is a simple version of Model 2 in Levin, Lin, and Chu (2002). LLC assume a general linear process for the time series of e_{it} and heterogeneity in e_{it} across i. We make the assumption of *iid* errors for analytic simplicity[1]. Also, we assume that $y_{i0} = 0$. This assumption could be relaxed by assuming that y_{i0} is $O_p(1)$ and satisfies some moment restrictions[2]. The panel z_{it} in this paper is assumed to have both a large number of time series observations and a large number of cross sections.

The null hypothesis we are interested in testing is that all series in the panel have a unit root:

$$\mathbb{H}_0 : \rho_i = 1 \text{ for all } i.$$

To test this hypothesis, LLC (2002) propose a test based on a modified $t-$ ratio statistic (see statistic t_δ^* for Model 2 defined on page 10 of LLC). They show that under the null hypothesis of a panel unit root, the test statistic has a standard normal limiting distribution. On the other hand, they relied on Monte Carlo simulations to investigate its power properties. In this paper, we compute analytically the asymptotic local power of the t - ratio test for a panel unit root.

For this, we consider the following specification for ρ_i :

$$\rho_i = 1 - \frac{\theta_i}{n^\kappa T} \text{ for some constant } \kappa > 0. \tag{2}$$

In this paper, for convenience, we assume the local-to-unity parameters θ_i are random and satisfy the following:

Assumption 1 *θ_i, $i = 1, ..., n$ are iid and independent of e_{it} for all i, t, and has mean μ_θ and variance σ_θ^2 on a non-negative bounded interval $[0, M_\theta]$, where $M_\theta \geq 0$. The support of θ_i is a subset of this interval.*

[1] At the expense of complicating the derivations, this assumption can be relaxed, and similar results to those in section 4 can be derived.

[2] See the discussion in section 6.2 of Moon, Perron, and Phillips (2007) on the difficulties in relaxing the initial condition restriction in a panel context.

Under Assumption 1, $\mathbb{H}_0$ is equivalent to

$$\mathbb{H}_0 : \mu_\theta = 0, \tag{3}$$

or $\theta_i = 0$ almost surely. [3] We consider a one-sided alternative hypothesis in which the autoregressive parameters can be heterogeneous across cross-sectional units:

$$\mathbb{H}_A : \mu_\theta > 0, \tag{4}$$

which implies that a positive fraction of cross-sectional units have a stationary autoregressive parameter ($\rho_i < 1$) under the alternative hypothesis.

We make the assumption that θ_i is *iid* to simplify the exposition and proofs. The results in the next two sections remain valid for local-to-unity parameters that are correlated (or even a deterministic sequence) as long as the cross-sectional averages of their powers converge to meaningful objects. Using independence allows us to easily use laws of large numbers in the cross-sectional dimension (as $n \to \infty$) to obtain our results. In the more general setting, the moments in the results below would be replaced by the limits of the appropriate cross-sectional averages. In particular, the homogeneous alternative in which all cross-sections have the same autoregressive coefficient considered by LLC is a special case of our framework with $\mu_\theta = \theta_i \neq 0$, $i = 1, ..., n$ and $\sigma_\theta^2 = 0$. Similarly, we restrict the distributions to be bounded but that is not necessary as long as enough moments exist. However, the assumption of independence between θ_i and e_{it} is crucial.

3 Test Statistics

Denote by Z, Z_{-1}, Y, Y_{-1}, and e the $(T \times n)$ matrices such that the $(t,i)^{th}$ element of each matrix is z_{it}, z_{it-1}, y_{it}, y_{it-1}, and e_{it}, respectively, where $t = 2, ..., T+1$ and $i = 1, ..., n$. Define $\alpha = (\alpha_1, ..., \alpha_n)'$, $l_T = (1, ..., 1)'$, and $\Theta = diag(\theta_1, ..., \theta_n)$. Using this notation, we can rewrite model (1) in matrix form as:

$$\begin{aligned} Z &= l_T \alpha' + Y, \\ Z_{-1} &= l_T \alpha' + Y_{-1}, \\ Y &= Y_{-1}\left(I_n - \frac{1}{n^\kappa T}\Theta\right) + e. \end{aligned} \tag{5}$$

We assume that an estimator $\hat{\sigma}^2$ of σ^2 that is consistent under the null and alternative hypotheses is available. An example of such an estimator is provided in Moon and Phillips (2004). They show that $\hat{\sigma}^2 - \sigma^2 = O_p\left(\frac{1}{\sqrt{T}}\max\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right)\right)$ where $\hat{\sigma}^2 = \frac{1}{nT} tr\left(\hat{e}'\hat{e}\right)$ and $\hat{e}$ is the matrix of residuals from a pooled autoregression on demeaned data[4]. In this paper, we assume for convenience:

[3] Note that this implies that the variance of θ, σ_θ^2, is 0 under the null hypothesis.

[4] See Lemma 2 of Moon and Phillips (2004).

Assumption 2 *$\hat{\sigma}^2 - \sigma^2 = o_p\left(\frac{1}{\sqrt{n}}\right)$ both under the null and alternative hypotheses.*

Finally, we make the following assumption on the behavior of the cross-sectional and time series dimensions:

Assumption 3 *n and T increase to infinity jointly with $\frac{\sqrt{n}}{T} \to 0$.*

In many applications, the fixed effect coefficients α_i in (5) are eliminated by taking out the time average from each cross-section, that is, by multiplying both sides of equation (5) by the projection matrix $Q_l = I_T - \frac{1}{T} l_T l_T'$. We use a tilde to denote a matrix multiplied by Q_l,for example, $\tilde{Z} = Q_l Z$. Then, we have

$$\tilde{Z} = \tilde{Y} = \tilde{Y}_{-1}\left(I_n - \frac{1}{n^{\kappa} T}\Theta\right) + \tilde{e}.$$

Considering that under the null hypothesis the autoregressive coefficient of y_{it-1} is one for all i (that is, $\theta_i = 0$ for all i), one may test the null hypothesis by using the pooled OLS estimator, say, $\hat{\rho}_{pool}$,

$$\hat{\rho}_{pool} = \left[tr\left(\tilde{Z}_{-1}\tilde{Z}_{-1}'\right)\right]^{-1} tr\left(\tilde{Z}_{-1}\tilde{Z}'\right).$$

To provide some intuition on the procedures below, it is instructive to consider the behavior of this pooled OLS estimator. By definition, we have

$$T\left(\hat{\rho}_{pool} - 1\right) = \left[\frac{1}{nT^2} tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right)\right]^{-1} \frac{1}{nT} tr\left(\tilde{Y}_{-1}\tilde{e}'\right).$$

It is well-known in the literature (*e.g.*, see Levin and Lin, 1992) that under the null hypothesis

$$\frac{1}{nT} tr\left(\tilde{Y}_{-1}\tilde{e}'\right) \to_p -\frac{1}{2}\sigma^2 \text{ and } \frac{1}{nT^2} tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right) \to_p \frac{1}{6}\sigma^2, \tag{6}$$

which leads to

$$T\left(\hat{\rho}_{pool} - 1\right) \to_p -3.$$

Notice that the numerator of $T\left(\hat{\rho}_{pool} - 1\right)$, $\frac{1}{nT} tr\left(\tilde{Y}_{-1}\tilde{e}'\right)$, has a non zero probability limit due to the correlation of the time demeaned y_{it} and the time demeaned e_{it}. This bias is related to the well-known incidental parameter problem in the conventional dynamic panel regression model.

The idea of the panel unit root test statistic proposed by LLC is to construct a t-ratio test based on a bias-adjusted pooled estimator. In this paper, we consider two different ways of bias correction and analyze their asymptotic properties in terms of local power. It turns out that the choice of method has an important impact on the power of resulting test. Define

$$\hat{\rho}_{pool}^{\#} = \left[tr\left(\tilde{Z}_{-1}\tilde{Z}_{-1}'\right)\right]^{-1}\left[tr\left(\tilde{Z}_{-1}\tilde{Z}'\right) + \frac{nT}{2}\hat{\sigma}^2\right]$$

and

$$\hat{\rho}^{+}_{pool} = \left[tr\left(\tilde{Z}_{-1}\tilde{Z}'_{-1}\right)\right]^{-1}\left[tr\left(\tilde{Z}_{-1}\tilde{Z}'\right)\right] + \frac{3}{T}.$$

The main difference between $\hat{\rho}^{\#}_{pool}$ and $\hat{\rho}^{+}_{pool}$ is that in $\hat{\rho}^{\#}_{pool}$, the numerator of $\hat{\rho}_{pool}$ is adjusted to ensure that it has mean 0, while in $\hat{\rho}^{+}_{pool}$, the bias of the entire estimator is subtracted. Bias-correction of the numerator was used by Hahn and Kuertseiner (2002). Notice by definition that

$$\hat{\rho}^{+}_{pool} = \hat{\rho}^{\#}_{pool} + \frac{3}{T}\left[\frac{1}{nT^2}tr\left(\tilde{Z}_{-1}\tilde{Z}'_{-1}\right)\right]^{-1}\left[\frac{1}{nT^2}tr\left(\tilde{Z}_{-1}\tilde{Z}'_{-1}\right) - \frac{1}{6}\hat{\sigma}^2\right], \tag{7}$$

and so, it is easy to see that

$$\hat{\rho}^{+}_{pool} = \hat{\rho}^{\#}_{pool} + o_p(1).$$

The asymptotics of $\hat{\rho}^{\#}_{pool}$ and $\hat{\rho}^{+}_{pool}$ under the null hypothesis are well known in the literature. For example, using the calculations in Baltagi and Kao (2000), the sequential limits as $T \to \infty$ and then $n \to \infty$, are:

$$\sqrt{n}T\left(\hat{\rho}^{\#}_{pool} - 1\right) \Rightarrow N(0, 3) \tag{8}$$

and

$$\sqrt{n}T\left(\hat{\rho}^{+}_{pool} - 1\right) \Rightarrow N\left(0, \frac{51}{5}\right). \tag{9}$$

Under Assumption 3, it is possible to show that the above two limits are also joint limits in the terminology of Phillips and Moon (1999).

Now in view of (6), (8), and (9), we can construct the following two different t- statistics,

$$t^{\#} = \sqrt{2}\frac{\sqrt{tr\left(\tilde{Z}_{-1}\tilde{Z}'_{-1}\right)}\left(\hat{\rho}^{\#}_{pool} - 1\right)}{\hat{\sigma}} \tag{10}$$

and

$$t^{+} = \sqrt{\frac{30}{51}}\frac{\sqrt{tr\left(\tilde{Z}_{-1}\tilde{Z}'_{-1}\right)}\left(\hat{\rho}^{+}_{pool} - 1\right)}{\hat{\sigma}}. \tag{11}$$

The $t^{\#}$ is the one that was analyzed in Section 3 of Moon and Perron (2004) and t^{+} is the one proposed by LLC.

Lemma 4 *Suppose that Assumptions 1, 2, and 3 hold. Under the null hypothesis, the following hold.*

(a) $$\begin{bmatrix} \sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Z}_{-1}\tilde{e}'\right) + \frac{1}{2}\hat{\sigma}^2\right] \\ \sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Z}_{-1}\tilde{Z}_{-1}\right) - \frac{1}{6}\hat{\sigma}^2\right] \end{bmatrix} \Rightarrow N\left(0, \begin{bmatrix} \frac{1}{12}\sigma^4 & 0 \\ 0 & \frac{1}{45}\sigma^4 \end{bmatrix}\right)$$

(b) $\frac{1}{nT^2}tr\left(\tilde{Z}_{-1}\tilde{Z}'_{-1}\right) \to_p \frac{1}{6}\sigma^2.$

From Lemma 4, we can deduce easily that under the null hypothesis, the weak limits of $t^{\#}$ and t^{+} are $N(0, 1)$. In what follows we investigate the asymptotic local power of the tests based on $t^{\#}$ and t^{+}.

4 Asymptotic Local Power

4.1 Asymptotic Local Power of a test based on $t^{\#}$

First, we investigate the limit of $t^{\#}$ under a local alternative. Before we introduce the main result, we first decompose $t^{\#}$ in (10) as

$$t^{\#}=\frac{\sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}\tilde{e}'\right)+\frac{1}{2}\hat{\sigma}^2-\frac{\sigma^2}{n^{\kappa}}\frac{\mu_\theta}{6}\right]-n^{1/2-\kappa}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\Theta\tilde{Y}'_{-1}\right)-\frac{\mu_\theta}{6}\sigma^2\right]}{\frac{\hat{\sigma}}{\sqrt{2}}\sqrt{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}'_{-1}\right)}}.$$

First, note that $E\left(\frac{1}{nT}tr\left(\tilde{Y}_{-1}\tilde{e}'\right)\right)-\frac{1}{2}\sigma^2-\frac{\sigma^2}{n^{\kappa}}\frac{\mu_\theta}{6}=O\left(n^{-2\kappa}\right)$ and $E\left(\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\Theta\tilde{Y}'_{-1}\right)\right)-\frac{\mu_\theta}{6}\sigma^2=O\left(n^{-\kappa}\right)$. One may then expect that both the first and the second terms of the numerator are of order $O_p\left(n^{1/2-2\kappa}\right)$. From this, one may deduce that $\kappa=\frac{1}{4}$ is required for significant local power of the $t^{\#}$ test.

In the following theorem we confirm this intuition rigorously. For this, we first consider local alternatives that shrink towards the null at a rate of $\frac{1}{n^{1/4}T}$. Let $\bar{z}_\alpha$ denote the $(1-\alpha)$-quantile of the standard normal distribution, *i.e.*, $P\left(Z\le-\bar{z}_\alpha\right)=\alpha$, where $Z\sim N\left(0,1\right)$, and $\Phi\left(x\right)$ denotes the CDF of Z.

Theorem 5 *Suppose that the local alternative in* (2) *shrinks towards the null at a rate of* $\frac{1}{n^{1/4}T}$ *(i.e.,* $\kappa=1/4$*). Suppose that Assumptions 1, 2, and 3 hold. Then*

(a) $t^{\#}\Rightarrow N\left(\frac{\mu_\theta^2+\sigma_\theta^2}{4\sqrt{3}},1\right)$;

(b) the asymptotic local power of the one-sided test of $\mathbb{H}_0$ *against* $\mathbb{H}_A$ *based on* $t^{\#}$ *is* $\Phi\left(-\frac{\mu_\theta^2+\sigma_\theta^2}{4\sqrt{3}}-\bar{z}_\alpha\right)$.

4.2 Asymptotic Local Power of a test based on t^{+}

In this subsection, we investigate the asymptotic local power of a one-sided test based on t^{+}. To do this, we narrow the local neighborhoods by assuming that the local alternative shrinks towards the null hypothesis at the faster rate of $\frac{1}{n^{1/2}T}$.

Assumption 6 $\kappa=1/2$.

From (7), (10), and (11), we can decompose t^{+} into two terms as:

$$t^{+}=\sqrt{\frac{15}{51}}t^{\#}+3\sqrt{\frac{30}{51}}\frac{\sqrt{n}\left\{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}'_{-1}\right)-\frac{1}{6}\hat{\sigma}^2\right\}}{\hat{\sigma}\sqrt{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}'_{-1}\right)}}. \quad (12)$$

Lemma 7 *Suppose that Assumptions 1, 2, 3, and 6 hold. Then, the following hold under both the null and alternative hypotheses.*

(a) $\left[\begin{array}{c} t^{\#} \\ \sqrt{n}\left\{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right)-\frac{1}{6}\hat{\sigma}^2\right\}\end{array}\right] \Rightarrow N\left(\left[\begin{array}{c} 0 \\ -\frac{\mu_\theta}{24}\sigma^2\end{array}\right],\left[\begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{45}\sigma^4\end{array}\right]\right)$

(b) $\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right) \to_p \frac{1}{6}\sigma^2$.

We can use Lemma 7 and Assumption 2 to derive that

$$t^{+} \Rightarrow N\left(-\frac{3}{2}\sqrt{\frac{5}{51}}\mu_\theta, 1\right)$$

under the local alternative hypothesis. From this, we can deduce that $-\bar{z}_\alpha$ is the asymptotic critical value of size α of the test, and that the asymptotic local power of the test is

$$\Phi\left(\frac{3}{2}\sqrt{\frac{5}{51}}\mu_\theta-\bar{z}_\alpha\right).$$

Summarizing this, we have the following theorem.

Theorem 8 *Suppose that Assumptions 1, 2, 3, and 6 hold. Then the following hold.*

(a) $t^{+} \Rightarrow N\left(-\frac{3}{2}\sqrt{\frac{5}{51}}\mu_\theta, 1\right)$ *under the local alternative;*

(b) the asymptotic local power of the one-sided test based on t^{+} *is* $\Phi\left(\frac{3}{2}\sqrt{\frac{5}{51}}\mu_\theta-\bar{z}_\alpha\right)$.

Remarks

(a) Unlike the test statistic $t^{\#}$, the LLC test statistic t^{+} has significant asymptotic power within a local neighborhood of order $n^{-1/2}T^{-1}$.

(b) Related to this, it is interesting to note from (8) and (9) that, under the null, bias-correction of the numerator only is more efficient in the sense of leading to an estimator, $\hat{\rho}^{\#}_{pool}$, with smaller variance. However, in Theorems 5 and 8, we find that the test based on the less efficient bias-corrected estimator $\hat{\rho}^{+}_{pool}$ has higher power than the test based on the more efficient estimator $\hat{\rho}^{\#}_{pool}$. To see why, notice from (7) that $\hat{\rho}^{+}_{pool}$ is the sum of $\hat{\rho}^{\#}_{pool}$ and a random component that is asymptotically independent of $\hat{\rho}^{\#}_{pool}$ under the null. Therefore, one deduces that the limit variance of $\hat{\rho}^{+}_{pool}$ is the sum of the limit variances of $\hat{\rho}^{\#}_{pool}$ and of the additional random component. This yields the efficiency of $\hat{\rho}^{\#}_{pool}$ over $\hat{\rho}^{+}_{pool}$ under the null. However, under the local alternative at the rate of order $n^{-1/2}T^{-1}$, the $t^{\#}$ statistic in (12), which is associated with $\hat{\rho}^{\#}_{pool}$, has the same limit distribution as the null limit distribution, while the distribution of the other term in (12) shifts in mean from the null distribution. Therefore, the power of t^+ is driven by the second term in (12), and the first term does not contribute asymptotically.

(c) From the above theorem, we notice that the asymptotic local power of the test based on t^+ depends only on the mean of θ_i, μ_θ and not on heterogeneity of the local alternative around this mean.

(d) Denote $V_o = \frac{\sqrt{n}}{\hat{\sigma}} \left\{ \frac{1}{nT^2} tr \left(\tilde{Z}_{-1} \tilde{Z}'_{-1} \right) - \frac{1}{6}\hat{\sigma}^2 \right\}$. Notice that the statistic V_o is a panel version of Sargan-Bhargava (1983) statistic for a unit root with the OLS-demeaned data. From the above discussion we can notice that the test statistic t^+ in (12) is a weighted average of two asymptotically independent statistics, $t^\#$ and V_o, and the asymptotic local power of t^+ is generated by V_o. Thus, one can deduce that the test rejecting the null for a small value of V_o has the same asymptotic local power as that of t^+.

(e) Notice that t^+ and V_o use the OLS-demeaned data. In the nonstationary time series literature, it is well known that GLS demeaning improves the power of unit root tests (*e.g.*, Elliott *et al*, 1996). In view of this, one may consider the following GLS version of Sargan-Bhargava statistic:

$$V_g = \frac{1}{\hat{\sigma}\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{1}{T^2} \sum_{t=1}^{T} (z_{it} - z_{i1})^2 - \frac{1}{2}\hat{\sigma}^2 \right\},$$

and reject the null hypothesis for a small value. Under the assumptions in Theorem 8, it is possible to show that under the local alternative with $\kappa = 1/2$,

$$\sqrt{3}V_g \Rightarrow N\left(-\frac{1}{\sqrt{3}}\mu_\theta, 1 \right)$$

and so, one can deduce easily that the power of this test is

$$\Phi\left(\frac{1}{\sqrt{3}}\mu_\theta - \bar{z}_\alpha \right).$$

(f) Recently, Moon, Perron, and Phillips (2007) derive the power envelope of the testing problem studied in this paper. According to them, the power envelope of the invariant panel unit root tests with fixed effects is $\Phi\left(\sqrt{\frac{\mu_\theta^2 + \sigma_\theta^2}{2}} - \bar{z}_\alpha \right)$. This shows that the power envelope is affected by the heterogeneity of the alternative hypothesis and the relative power loss of the LLC test increases as the alternative hypothesis becomes more heterogeneous.

(g) In addition, Moon, Perron, and Phillips (2007) propose a common point optimal invariant panel unit root test (which we call the MPP test) and show that its asymptotic local power is $\Phi\left(\frac{\mu_\theta}{\sqrt{2}} - \bar{z}_\alpha \right)$ which is higher than the power of t^+.

(h) If we combine our results, we can order the three tests discussed here in decreasing order of power as MPP, V_g, and t^+ (or V_o).

5 Simulation results

In this section, we document the usefulness of the asymptotic analysis of the previous sections in explaining the finite-sample behavior of the tests considered. For this purpose. we will use model (1) with e_{it} generated as $iid\ N(0,1)$ and α_i set to zero.

We consider six alternatives in total, four that are local to the null hypothesis and two that fix the autoregressive parameters. The four alternatives that fix the distribution of the local-to-unity parameters provide an easy gauge of the accuracy of the asymptotic analysis, but the two alternatives that fix the autoregressive parameters are more intuitive. The alternatives come in three pairs. The first alternative in the pair is heterogeneous, while the second one is homogeneous. Our theoretical results suggest that all tests should have the same power within a pair. However, the power envelope should be higher for the heterogeneous alternative.

The four local alternatives are defined as:

$$\begin{aligned}\theta_i &\sim U[0,2]\\ \theta_i &= 1 \quad \forall i\\ \theta_i &\sim U[0,8]\\ \theta_i &= 4 \quad \forall i\end{aligned}$$

The first and third alternatives are heterogeneous, while the second and fourth are homogeneous with the same mean as the previous heterogeneous alternative. The autoregressive parameters are defined from these distributions by the $n^{-1/2}T^{-1}$ rate that is appropriate for all tests except for the $t^{\#}$ test. If our asymptotics provide reliable predictions about the finite-sample behavior of the statistics, we should see the same power for all choices of n and T for a given alternative. Since the rate we use to define the autoregressive parameter is too fast, the $t^{\#}$ test should have no power beyond size (the asymptotic distribution is the same under the null and alternative hypotheses). Finally, the two fixed alternatives are $\rho_i \sim U[0.98, 1]$ and $\rho_i = 0.99$ for all i. Thus, the mean of the autoregressive parameter is the same in both of these cases as well.

We consider all the tests discussed above, that is the $t^{\#}$ test that bias-corrects the numerator of $\hat{\rho}_{pool}$ only, the t^+ test that bias-corrects $\hat{\rho}_{pool}$, the panel version of the Sargan-Bhargava statistic with GLS demeaning (V_g), and the common point-optimal test of Moon, Perron, and Phillips (2007) with $c = 1$ (MPP). Finally, for the four local alternatives, we report the power envelope, obtained by using the actual θ_i's in computing the likelihood ratio statistic of Moon, Perron, and Phillips. We report results for the same choices of n and T as LLC, that is $n = 10$, 25, 50, 100, and 250 and $T = 25$, 50, 100, and 250. Each experiment was repeated 10,000 times, and the level of the tests was set at 5%.

The results from all experiments are reported in tables 1 (size) and 2-7 (size-adjusted power). The tables report the results for the tests in the order in which our theory predicts power should be increasing, that is $t^{\#}$, t^{+}, V_g, MPP, and the power envelope. The power level suggested by our theory for each test can be found at the bottom of tables 2-5.

The main findings in table 1 regarding size are:

- the $t^{\#}$ test tends to overreject the null hypothesis for small values of T. There is also an increase in size distortions for larger values of n relative to T;
- the t^{+} and MPP tests control size very well;
- the V_g test underrejects quite severely, but this tendency diminishes as T increases.

For the local alternatives in which the distribution of the local-to-unity parameters is fixed, the main results are:

- power is quite flat as we increase n and T for all tests, though we can see a small increase among the smaller values of n;
- the power level and relative ranking of each test is well predicted by our theory;
- there is little difference between the homogeneous and heterogeneous alternatives. Note that the MPP test is optimal under the homogeneous alternatives, and its power is quite close to the power envelope;
- the finite-sample behavior of the power envelope for small n is much below both the theoretical results and the power of some of the other tests;

Finally, for the fixed alternatives with mean autoregressive parameter equal to 0.99, our main findings are:

- as predicted by our theory, power increases faster along the T dimension than along the n dimension for all tests (with the obvious exception of the $t^{\#}$ test);
- as expected, the $t^{\#}$ test has very little power and should not be used in practice. Its power goes down with n and T due to the shift to the right of the distribution;
- despite our theoretical results, there is somewhat of a power loss against a heterogeneous alternative in finite samples;
- the power envelope is higher in the heterogeneous case and thus the power loss relative to the power envelope for the tests considered here is more important than in the homogeneous case.

Overall, it appears that our theoretical results are quite useful to predict the finite-sample behavior of the various tests we have considered. In particular, they predict very well that the $t^{\#}$ test should not be used in practice and that the common point optimal test if Moon, Perron, and Phillips (2007) achieves power that is very close to the power envelope.

6 Conclusion

In this paper we derive the asymptotic local power of two tests for a panel unit root based on bias-corrected pooled OLS estimators. An interesting by-product of our analysis is the finding that the method used to correct for the bias of the pooled OLS estimator is not innocuous and has very important consequences in terms of the asymptotic local power of the resulting tests. We find that the test that is equivalent to Levin, Lin, and Chu (2002)'s t-test has significant local power in $n^{-1/2}T^{-1}$ neighborhoods of a unit root and that this power depends on the first moment of the local-to-unity parameter only. On the other hand, bias correction of the numerator of the estimator only results in a test which has power against local alternatives that shrink towards the null hypothesis at the slower $n^{-1/4}T^{-1}$ rate.

Many interesting extensions of the present work are possible. Most importantly, it would be interesting to derive the asymptotic local power of other tests for panel unit roots such as those of Im, Pesaran, and Shin (2002) or Maddala and Wu (1999) which *may* be better adapted to the heterogeneous alternative that we have considered. Nevertheless, as mentioned earlier in the introduction, these tests are inadmissible by the results of Bowman (2002).

7 Appendix

7.1 Preliminary Results

We denote by $y_{it}(0)$ the panel data y_{it} generated by model (1) with $\rho_i = 1$, that is, $\theta_i = 0$ for all i. We also define $Y(0)$ and $Y_{-1}(0)$, respectively, in similar fashion to Y and Y_{-1}. Notice by definition that

$$\begin{aligned} y_{it-1} - y_{it-1}(0) &= \sum_{p=1}^{t-1} \left(\rho_i^{t-p} - 1\right) e_{ip} = \sum_{p=1}^{t-1} \left[\sum_{l=1}^{t-p} \binom{t-p}{l} \left(\frac{-\theta_i}{n^{\kappa} T} \right)^{l} \right] e_{ip} \text{ for } t \geq 2 \\ &= 0 \text{ for } t = 1. \end{aligned} \tag{13}$$

We also denote $\tilde{y}_{it-1}$, $\tilde{y}_{it-1}(0)$, and $\tilde{e}_{it}$ to be the $(i,t)^{th}$ element of matrices $\tilde{Y}_{-1}$, $\tilde{Y}_{-1}(0)$, and $\tilde{e}$.

Lemma 9 *Suppose that Assumptions 1, 2, and 3 hold. Both under the null and alternative hypotheses with $\kappa = 1/4$ and $1/2$, the following hold.*

(a) $\begin{bmatrix} \sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}(0)\tilde{e}'\right)+\frac{1}{2}\sigma^2\right] \\ \sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\tilde{Y}_{-1}(0)'\right)-\frac{1}{6}\sigma^2\right] \end{bmatrix} \Rightarrow N\left(0,\begin{bmatrix} \frac{1}{12}\sigma^4 & 0 \\ 0 & \frac{1}{45}\sigma^4 \end{bmatrix}\right).$

(b) $\sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\Theta\tilde{Y}_{-1}(0)'\right)-\frac{1}{6}\mu_\theta\sigma^2\right]=O_p(1).$

(c) $\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right)\to_p\frac{1}{6}\sigma^2.$

Proof

Part (a). Notice by direct calculations that

$$E\left[\frac{1}{T}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)\tilde{e}_{it}\right] = -\frac{1}{2}\sigma^2+O\left(\frac{1}{T}\right),$$

$$E\left[\frac{1}{T^2}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)^2\right] = \frac{1}{6}\sigma^2+O\left(\frac{1}{T}\right),$$

and

$$Var\left[\frac{1}{T}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)\tilde{e}_{it}\right] = \frac{1}{12}\sigma^4+O\left(\frac{1}{T}\right),$$

$$Var\left[\frac{1}{T^2}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)^2\right] = \frac{1}{45}\sigma^4+O\left(\frac{1}{T}\right),$$

$$cov\left[\frac{1}{T}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)\tilde{e}_{it},\frac{1}{T^2}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)^2\right] = O\left(\frac{1}{T}\right).$$

Then, by applying the central limit theorem in Phillips and Moon (1999), we can deduce that

$$\begin{aligned}&\begin{bmatrix} \sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}(0)\tilde{e}'\right)+\frac{1}{2}\sigma^2\right] \\ \sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\tilde{Y}_{-1}(0)'\right)-\frac{1}{6}\sigma^2\right] \end{bmatrix} \\ = \ & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{bmatrix} \frac{1}{T}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)\tilde{e}_{it}-E\left(\frac{1}{T}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)\tilde{e}_{it}\right) \\ \frac{1}{T^2}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)^2-E\left(\frac{1}{T^2}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)^2\right) \end{bmatrix}+O\left(\frac{\sqrt{n}}{T}\right) \\ \Rightarrow \ & N\left(0,\begin{bmatrix} \frac{1}{12}\sigma^4 & 0 \\ 0 & \frac{1}{45}\sigma^4 \end{bmatrix}\right),\end{aligned}$$

as required. ■

Part (b) holds because $Var\left\{\sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\Theta\tilde{Y}_{-1}(0)'\right)-\frac{1}{6}\mu_\theta\sigma^2\right]\right\}=Var\left[\theta_i\frac{1}{T^2}\sum_{t=2}^{T+1}\tilde{y}_{it-1}(0)^2\right]<M$ for some finite constant M. ■

Part (c) holds by arguments similar to those used in the proof of Lemma 5 of Moon and Perron (2004) with $k=0$, $\eta=0$, $1/4$, and $1/2$, $b_{k,nT}=-\frac{1}{2}\sigma^2$, $Q_{\beta^0}=I_n$, and $h_k(r,s)=1$. Thus, we omit the proof. ■

Lemma 10 *Assume that the local alternative in (2) shrinks towards the null at a rate of* $\frac{1}{n^{1/4}T}$ *(i.e.,* $\kappa = 1/4$*). Suppose that Assumptions 1, 2, and 3 hold. Then, the following hold under the local alternative hypothesis.*

(a) $\frac{1}{n^{3/4}T^2} tr\left\{\left(\tilde{Y}_{-1} - \tilde{Y}_{-1}(0)\right)\Theta\left(\tilde{Y}_{-1} - \tilde{Y}_{-1}(0)\right)'\right\} = o_p(1)$.

(b) $\frac{1}{n^{3/4}T^2} tr\left\{\left(\tilde{Y}_{-1} - \tilde{Y}_{-1}(0)\right)\Theta\tilde{Y}_{-1}(0)'\right\} \to_p -\frac{(\mu_\theta^2+\sigma_\theta^2)\sigma^2}{24}$.

(c) $\sqrt{n}\left[\frac{1}{nT} tr\left\{\left(\tilde{Y}_{-1} - \tilde{Y}_{-1}(0)\right)\tilde{e}'\right\} - \frac{\sigma^2}{n^{1/4}}\frac{\mu_\theta}{6}\right] \to_p -\frac{(\mu_\theta^2+\sigma_\theta^2)\sigma^2}{24}$.

Proof

Part (a) holds by arguments similar to those used in the proof of Lemma 5 of Moon and Perron (2004) with $k = 0$, $\eta = 1/4$, $b_{k,nT} = -\frac{1}{2}\sigma^2$, $Q_{\beta^0} = I_n$, and $h_k(r,s) = 1$. Thus, we omit the proof of this result and provide only the proofs of Parts (b) and (c).

Part (b). Notice that

$$
\begin{aligned}
& \frac{1}{n^{3/4}T^2} tr\left\{\left(\tilde{Y}_{-1} - \tilde{Y}_{-1}(0)\right)\Theta\tilde{Y}_{-1}(0)'\right\} \\
= & \frac{1}{n^{3/4}T^2} tr\left\{(Y_{-1} - Y_{-1}(0))\Theta Y_{-1}(0)'\right\} - \frac{1}{n^{3/4}T^3} tr\left\{l_T'(Y_{-1} - Y_{-1}(0))\Theta Y_{-1}(0)' l_T\right\} \\
= & \frac{1}{n^{3/4}T^2}\sum_{i=1}^{n}\theta_i\sum_{t=2}^{T+1}(y_{it-1} - y_{it-1}(0))\,y_{it-1}(0) - \frac{1}{n^{3/4}T^3}\sum_{i=1}^{n}\theta_i\sum_{t=2}^{T+1}(y_{it-1} - y_{it-1}(0))\sum_{s=2}^{T+1}y_{is-1}(0) \\
= & I_b - II_b, \text{ say.}
\end{aligned}
$$

By (13) and the law of large numbers, we have

$$
\begin{aligned}
I_b &= \frac{1}{n^{3/4}T^2}\sum_{i=1}^{n}\theta_i\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/4}T}\right)^l\right]e_{ip}\sum_{s=1}^{t-1}e_{is} \\
&= -\frac{1}{n}\sum_{i=1}^{n}\theta_i^2\frac{1}{T^2}\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\sum_{s=1}^{t-1}\left(\frac{t-p}{T}\right)e_{ip}e_{is} + o_p(1) \\
&\to_p -E\left(\theta_i^2\right)\sigma^2\int_0^1\int_0^r (r-p)\,dpdr = -\frac{1}{6}\sigma^2\left(\mu_\theta^2 + \sigma_\theta^2\right)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
II_b &= \frac{1}{n^{3/4}T^3}\sum_{i=1}^{n}\theta_i\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/4}T}\right)^l\right]e_{ip}\sum_{s=2}^{T+1}\sum_{q=1}^{s-1}e_{iq} \\
&= -\frac{1}{n}\sum_{i=1}^{n}\theta_i^2\frac{1}{T^3}\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\left(\frac{t-p}{T}\right)e_{ip}\sum_{s=2}^{T+1}\sum_{q=1}^{s-1}e_{iq} + o_p(1) \\
&\to_p -E\left(\theta_i^2\right)\sigma^2\int_0^1\int_0^1\int_0^{\min(r,s)}(r-p)\,dpdsdr = -\frac{1}{8}\sigma^2\left(\mu_\theta^2 + \sigma_\theta^2\right).
\end{aligned}
$$

Combining the limits of I_b and II_b, we have the required result for Part (e),

$$\frac{1}{n^{3/4}T^2} tr\left\{\left(\tilde{Y}_{-1} - \tilde{Y}_{-1}(0)\right)\Theta\tilde{Y}_{-1}(0)'\right\} \to_p -\frac{1}{24}\sigma^2\left(\mu_\theta^2 + \sigma_\theta^2\right). \blacksquare$$

Part (c). By definition,

$$\begin{aligned}
&\sqrt{n}\left[\frac{1}{nT} tr\left\{\left(\tilde{Y}_{-1} - \tilde{Y}_{-1}(0)\right)\tilde{e}'\right\} - \frac{\sigma^2}{n^{1/4}}\frac{\mu_\theta}{6}\right] \\
= \quad & \frac{1}{\sqrt{n}T} tr\left[e'\left(Y_{-1} - Y_{-1}(0)\right)\right] - \sqrt{n}\left[\frac{1}{nT^2} tr\left(e' l_T l_T'\left(Y_{-1} - Y_{-1}(0)\right)\right) + \frac{\sigma^2}{n^{1/4}}\frac{\mu_\theta}{6}\right] = I_c - II_c, \text{ say.}
\end{aligned}$$

First, from (13) we have for some constant $M < \infty$,

$$\begin{aligned}
E\left(I_c^2\right) &= \frac{1}{nT^2} E\left[\sum_{i=1}^{n}\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/4}T}\right)^l\right] e_{ip}e_{it}\right]^2 \\
&= \frac{1}{nT^2} E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\sum_{s=2}^{T+1}\sum_{q=1}^{s-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/4}T}\right)^l\right]\left[\sum_{k=1}^{s-q}\binom{s-q}{k}\left(\frac{-\theta_i}{n^{1/4}T}\right)^k\right] e_{ip}e_{it}e_{jq}e_{js}\right] \\
&= \frac{1}{T^2}\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\sum_{s=2}^{T+1}\sum_{q=1}^{s-1} E\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/4}T}\right)^l\right]\left[\sum_{k=1}^{s-q}\binom{s-q}{k}\left(\frac{-\theta_i}{n^{1/4}T}\right)^k\right] E\left(e_{ip}e_{it}e_{iq}e_{is}\right) \\
&\le \frac{1}{n^{1/2}}\frac{M}{T^2}\sum_{t=2}^{T+1}\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{1}{T}\right)^l\right]\left[\sum_{k=1}^{t-p}\binom{t-p}{k}\left(\frac{1}{T}\right)^k\right] = O\left(\frac{1}{n^{1/2}}\right),
\end{aligned}$$

where the third equality holds because for $i \neq j$, $E\left(e_{ip}e_{it}e_{jq}e_{js}\right) = E\left(e_{ip}e_{it}\right)E\left(e_{jq}e_{js}\right) = 0$ due to independence of e_{it} across i and t, and the inequality holds because $E\left(e_{ip}e_{it}e_{iq}e_{is}\right) = 0$ if $t \neq s$ or $p \neq q$ and the support θ_i is bounded.

This yields

$$I_c = o_p(1).$$

Next,

$$II_c = \sqrt{n}\left[\frac{1}{nT^2} tr\left(e' l_T l_T'\left(Y_{-1} - Y_{-1}(0)\right)\right) + \frac{\sigma^2\mu_\theta}{n^{1/4}}\frac{1}{T^2}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)\right] + o_p(1),$$

because $\frac{1}{T^2}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) - \int_0^1\int_0^r (r-s)\,dsdr\left(=\frac{1}{6}\right) = O\left(\frac{1}{T}\right)$. Notice from (13) that

$$\begin{aligned}
&\frac{1}{nT^2}tr\left(e'l_Tl_T'\left(Y_{-1}-Y_{-1}(0)\right)\right)\\
=\;&\frac{1}{nT^2}\sum_{i=1}^{n}\left(\sum_{p=2}^{T+1}e_{ip}\right)\left(\sum_{t=1}^{T+1}\left(y_{it-1}-y_{it-1}(0)\right)\right)\\
=\;&-\frac{1}{n^{5/4}}\sum_{i=1}^{n}\theta_i\left(\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip}\right)\\
&+\frac{1}{2n^{3/2}}\sum_{i=1}^{n}\theta_i^2\left(\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=3}^{T+1}\sum_{s=1}^{t-2}\left(\frac{t-s}{T}\right)\left(\frac{t-s-1}{T}\right)e_{is}e_{ip}\right)+o_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Notice that

$$E\left[\theta_i\left(\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip}\right)\right]=\sigma^2\mu_\theta\frac{1}{T^2}\sum_{t=3}^{T+1}\sum_{s=2}^{t-1}\left(\frac{t-s-1}{T}\right).$$

Then, we have

$$\begin{aligned}
II_c\;=\;&-\frac{1}{n^{3/4}}\sum_{i=1}^{n}\left\{\theta_i\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip}-E\left(\theta_i\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip}\right)\right\}\\
&+\frac{1}{2n}\sum_{i=1}^{n}\theta_i^2\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=3}^{T+1}\sum_{s=1}^{t-2}\left(\frac{t-s}{T}\right)\left(\frac{t-s-1}{T}\right)e_{is}e_{ip}+o_p(1)\\
=\;&O_p\left(\frac{1}{n^{1/4}}\right)+\frac{E\left(\theta_i^2\right)}{2}\sigma^2\int_0^1\int_0^r(r-s)^2\,dsdr+o_p(1).
\end{aligned}$$

So,

$$II_c\to_p\frac{\left(\mu_\theta^2+\sigma_\theta^2\right)\sigma^2}{24}.$$

Combining the limits of I_c and II_c yields the desired result for Part (c). ■

Lemma 11 *Assume that the local alternative in* (2) *shrinks towards the null at a rate of* $\frac{1}{n^{1/4}T}$ *(i.e.,* $\kappa=1/4$*). Suppose that Assumptions 1, 2, and 3 hold. Then, the following hold under the local alternative hypothesis.*

(a) $\sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}\tilde{e}'\right)+\frac{1}{2}\hat{\sigma}^2-\frac{\sigma^2}{n^{1/4}}\frac{\mu_\theta}{6}\right]\Rightarrow N\left(-\frac{\left(\mu_\theta^2+\sigma_\theta^2\right)\sigma^2}{24},\frac{1}{12}\sigma^4\right)$.

(b) $n^{1/4}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\Theta\tilde{Y}_{-1}'\right)-\frac{\mu_\theta}{6}\sigma^2\right]\to_p-\frac{\left(\mu_\theta^2+\sigma_\theta^2\right)\sigma^2}{12}$.

(c) $\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right)\to_p\frac{1}{6}\sigma^2$.

Proof
Part (a). By Lemma 4(a), Assumption 2, and Lemma 10(c),

$$\begin{aligned}
& \sqrt{n}\left[\frac{1}{nT}tr\left\{\tilde{Y}_{-1}\tilde{e}'\right\}+\frac{1}{2}\hat{\sigma}^2-\frac{\sigma^2}{n^{1/4}}\frac{\mu_\theta}{6}\right] \\
= & \sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}(0)\,\tilde{e}'\right)+\frac{1}{2}\sigma^2\right]+\frac{1}{2}\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \\
& +\sqrt{n}\left[\frac{1}{nT}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{e}'\right\}-\frac{\sigma^2}{n^{1/4}}\frac{\mu_\theta}{6}\right] \\
\Rightarrow & N\left(-\frac{\left(\mu_\theta^2+\sigma_\theta^2\right)\sigma^2}{24},\frac{1}{12}\sigma^4\right),
\end{aligned}$$

as required for Part (a). ■

Part (b). By Lemma 10(a), (b), and Lemma 9(b), we have

$$\begin{aligned}
& n^{1/4}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\Theta\tilde{Y}_{-1}'\right)-\frac{\mu_\theta}{6}\sigma^2\right] \\
= & \frac{1}{n^{3/4}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\Theta\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)'\right\}+2\frac{1}{n^{3/4}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\Theta\tilde{Y}_{-1}(0)'\right\} \\
& +\frac{1}{n^{1/4}}\sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\,\Theta\tilde{Y}_{-1}(0)'\right)-\frac{\mu_\theta}{6}\sigma^2\right] \\
\to & _p-\frac{\left(\mu_\theta^2+\sigma_\theta^2\right)\sigma^2}{12},
\end{aligned}$$

as required for Part (b). ■

Part (c) holds by Lemma 9(c). ■

Lemma 12 *Suppose that Assumptions 1, 2, and 3 hold. Under Assumption 6, the following hold under the local alternative hypothesis.*

(a) $\frac{1}{n^{1/2}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)'\right\}=o_p(1)$.

(b) $\frac{1}{n^{1/2}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{Y}_{-1}(0)'\right\}\to_p-\frac{\mu_\theta}{24}\sigma^2$.

(c) $\sqrt{n}\left[\frac{1}{nT}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{e}'\right\}-\frac{\sigma^2}{n^{1/2}}\frac{\mu_\theta}{6}\right]=o_p(1)$.

Proof

Part (a). Using (13) , we write

$$
\begin{aligned}
& \frac{1}{n^{1/2}T^2} tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)'\right\} \\
= \; & \frac{1}{n^{1/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left(y_{it-1}-y_{it-1}(0)\right)^2-\frac{1}{n^{1/2}T^3}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left(y_{it-1}-y_{it-1}(0)\right)\sum_{s=2}^{T+1}\left(y_{is-1}-y_{is-1}(0)\right) \\
= \; & \frac{1}{n^{1/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/2}T}\right)^{l}\right]e_{ip}\right\}\left\{\sum_{q=1}^{t-1}\left[\sum_{m=1}^{t-q}\binom{t-q}{m}\left(\frac{-\theta_i}{n^{1/2}T}\right)^{m}\right]e_{iq}\right\} \\
& -\frac{1}{n^{1/2}T^3}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/2}T}\right)^{l}\right]e_{ip}\right\}\sum_{s=2}^{T+1}\left\{\sum_{q=1}^{s-1}\left[\sum_{m=1}^{s-q}\binom{s-q}{m}\left(\frac{-\theta_i}{n^{1/2}T}\right)^{m}\right]e_{iq}\right\} \\
= \; & I_a - II_a, \text{ say.}
\end{aligned}
$$

Since e_{ip} are iid with mean zero and finite fourth moments,

$$
I_a=\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{\theta_i^2}{T^2}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\frac{t-p}{T}e_{ip}\right\}\left\{\sum_{q=1}^{t-1}\frac{t-q}{T}e_{iq}\right\}+O_p\left(\frac{1}{n}\right)=O_p\left(\frac{1}{\sqrt{n}}\right),
$$

and similarly

$$
II_a=\frac{1}{n^{3/2}}\sum_{i=1}^{n}\frac{\theta_i^2}{T^3}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\frac{t-p}{T}e_{ip}\right\}\sum_{s=2}^{T+1}\left\{\sum_{q=1}^{s-1}\frac{s-q}{T}e_{iq}\right\}+O_p\left(\frac{1}{n}\right)=O_p\left(\frac{1}{\sqrt{n}}\right).
$$

Therefore,

$$
\frac{1}{n^{1/2}T^2} tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)'\right\}=o_p(1),
$$

as required. ■

Part (b). Using (13) , we write

$$
\begin{aligned}
& \frac{1}{n^{1/2}T^2} tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{Y}_{-1}(0)'\right\} \\
= \; & \frac{1}{n^{1/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left(y_{it-1}-y_{it-1}(0)\right)y_{it-1}(0)-\frac{1}{n^{1/2}T^3}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left(y_{it-1}-y_{it-1}(0)\right)\sum_{s=2}^{T+1}y_{is-1}(0) \\
= \; & \frac{1}{n^{1/2}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/2}T}\right)^{l}\right]e_{ip}\right\}\left\{\sum_{q=1}^{t-1}e_{iq}\right\} \\
& -\frac{1}{n^{1/2}T^3}\sum_{i=1}^{n}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\left[\sum_{l=1}^{t-p}\binom{t-p}{l}\left(\frac{-\theta_i}{n^{1/2}T}\right)^{l}\right]e_{ip}\right\}\sum_{s=2}^{T+1}\left\{\sum_{q=1}^{s-1}e_{iq}\right\} \\
= \; & I_b - II_b, \text{ say.}
\end{aligned}
$$

By the law of large numbers as in Phillips and Moon (1999), we have

$$\begin{aligned} I_b &= -\frac{1}{n}\sum_{i=1}^{n}\theta_i\left[\frac{1}{T^2}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\frac{t-p}{T}e_{ip}\right\}\left\{\sum_{q=1}^{t-1}e_{iq}\right\}\right]+o_p(1) \\ &\to {}_p -E(\theta_i)\sigma^2\int_0^1\int_0^r (r-p)\,dpdr = -\frac{1}{6}\mu_\theta\sigma^2. \end{aligned}$$

Similarly,

$$\begin{aligned} II_b &= -\frac{1}{n}\sum_{i=1}^{n}\theta_i\left[\frac{1}{T^3}\sum_{t=2}^{T+1}\left\{\sum_{p=1}^{t-1}\frac{t-p}{T}e_{ip}\right\}\sum_{s=2}^{T+1}\left\{\sum_{q=1}^{s-1}e_{iq}\right\}\right]+o_p(1) \\ &\to {}_p -E(\theta_i)\sigma^2\int_0^1\int_0^1\int_0^{\min(r,s)} (r-p)\,dpdsdr = -\frac{1}{8}\mu_\theta\sigma^2. \end{aligned}$$

Combining the limits of I_b and II_b, we have the required result for Part (b). ■

Part (c). By definition,

$$\begin{aligned} &\frac{1}{\sqrt{n}T}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{e}'\right\}-\sigma^2\frac{\mu_\theta}{6} \\ = &\frac{1}{\sqrt{n}T}tr\left[e'\left(Y_{-1}-Y_{-1}(0)\right)\right]-\left[\frac{1}{\sqrt{n}T^2}tr\left(e'l_Tl_T'\left(Y_{-1}-Y_{-1}(0)\right)\right)+\sigma^2\frac{\mu_\theta}{6}\right]=I_c-II_c, \text{ say.} \end{aligned}$$

Notice by similar arguments used in the proof of $I_c = o_p(1)$ in Lemma 10(c), we have

$$I_c = o_p(1).$$

Next,

$$II_c = \frac{1}{\sqrt{n}T^2}tr\left(e'l_Tl_T'\left(Y_{-1}-Y_{-1}(0)\right)\right)+\sigma^2\mu_\theta\frac{1}{T^2}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)+o(1),$$

because $\frac{1}{T^2}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)-\int_0^1\int_0^r(r-s)\,dsdr\left(=\frac{1}{6}\right)=O\left(\frac{1}{T}\right)$. Notice from (13) that

$$\begin{aligned} &\frac{1}{\sqrt{n}T^2}tr\left(e'l_Tl_T'\left(Y_{-1}-Y_{-1}(0)\right)\right) \\ = &\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\left(\sum_{p=2}^{T+1}e_{ip}\right)\left(\sum_{t=2}^{T+1}\left(y_{it-1}-y_{it-1}(0)\right)\right) \\ = &-\frac{1}{n}\sum_{i=1}^{n}\theta_i\left(\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip}\right)+O_p\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

Notice that

$$E\left[\theta_i\left(\frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip}\right)\right]=\sigma^2\mu_\theta\frac{1}{T^2}\sum_{t=3}^{T+1}\sum_{s=2}^{t-1}\left(\frac{t-s}{T}\right).$$

Then, we have

$$\begin{aligned} II_c &= -\frac{1}{n}\sum_{i=1}^{n}\left\{ \begin{array}{c} \theta_i \frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip} \\ -E\left(\theta_i \frac{1}{T^2}\sum_{p=2}^{T+1}\sum_{t=2}^{T+1}\sum_{s=1}^{t-1}\left(\frac{t-s}{T}\right)e_{is}e_{ip}\right) \end{array}\right\} + o_p(1) \\ &= o_p(1) \end{aligned}$$

by the law of large numbers. Therefore we have the required results for Part (c). ■

7.2 Proof of Main Results

Proof of Lemma 4

The lemma holds by Lemma 9 and Assumption 2. ■

Proof of Theorem 5

By the definition of $t^{\#}$ in (10), we decompose

$$\begin{aligned} t^{\#} &= \frac{\sqrt{n}\left[\frac{1}{nT}tr\left\{\tilde{Y}_{-1}\left(-\frac{1}{n^{1/4}T}\tilde{Y}_{-1}\Theta+\tilde{e}\right)'\right\}+\frac{1}{2}\hat{\sigma}^2\right]}{\frac{\hat{\sigma}}{\sqrt{2}}\sqrt{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}'_{-1}\right)}} \\ &= \frac{\sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}\tilde{e}'\right)+\frac{1}{2}\hat{\sigma}^2-\frac{\sigma^2}{n^{1/4}}\frac{\mu_\theta}{6}\right]-n^{1/4}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\Theta\tilde{Y}'_{-1}\right)-\frac{\mu_\theta}{6}\sigma^2\right]}{\frac{\hat{\sigma}}{\sqrt{2}}\sqrt{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}'_{-1}\right)}}. \end{aligned}$$

The required results for the theorem follow by Lemma 11, Assumption 2, and Slutzky theorem. ■

Proof of Lemma 7

Part (a). When $\kappa = 1/2$, notice that

$$t^{\#} = \sqrt{2}\frac{\sqrt{n}\left[\frac{1}{nT}tr\left\{\tilde{Y}_{-1}\left(-\frac{1}{n^{1/2}T}\tilde{Y}_{-1}\Theta+\tilde{e}\right)'\right\}+\frac{1}{2}\hat{\sigma}^2\right]}{\hat{\sigma}\sqrt{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}'_{-1}\right)}}.$$

The numerator of $t^{\#}$ can be further decomposed as

$$\begin{aligned}
&\sqrt{n}\left[\frac{1}{nT}tr\left\{\tilde{Y}_{-1}\left(-\frac{1}{n^{1/2}T}\tilde{Y}_{-1}\Theta+\tilde{e}\right)'\right\}+\frac{1}{2}\hat{\sigma}^2\right] \\
= \; & \sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}(0)\,\tilde{e}'\right)+\frac{1}{2}\sigma^2\right]+\frac{1}{2}\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \\
& +\left[\frac{1}{\sqrt{n}T}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{e}'\right\}-\sigma^2\frac{\mu_\theta}{6}\right]-\frac{1}{nT^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\Theta\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)'\right\} \\
& -2\frac{1}{nT^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\Theta\tilde{Y}_{-1}(0)'\right\}-\frac{1}{n^{1/2}}\sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\,\Theta\tilde{Y}_{-1}(0)'\right)-\frac{\mu_\theta}{6}\sigma^2\right] \\
= \; & \sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}(0)\,\tilde{e}'\right)+\frac{1}{2}\sigma^2\right]+o_p(1), \qquad (14)
\end{aligned}$$

where the last line holds by Assumption 2, Lemma 12, and Lemma 9(b). Therefore, together with Lemma 11(c) and Assumption 2, we have

$$t^{\#}=\frac{\sqrt{12}}{\sigma^2}\sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}(0)\,\tilde{e}'\right)+\frac{1}{2}\sigma^2\right]+o_p(1). \qquad (15)$$

Next, notice that

$$\begin{aligned}
&\sqrt{n}\left\{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right)-\frac{1}{6}\hat{\sigma}^2\right\} \\
= \; & \sqrt{n}\left\{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right)-\frac{1}{6}\sigma^2\right\}-\frac{1}{6}\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \\
= \; & \sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\,\tilde{Y}_{-1}(0)'\right)-\frac{1}{6}\sigma^2\right]+2\frac{1}{n^{1/2}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{Y}_{-1}(0)'\right\} \\
& +\frac{1}{n^{1/2}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)'\right\}-\frac{1}{6}\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \\
= \; & \sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\,\tilde{Y}_{-1}(0)'\right)-\frac{1}{6}\sigma^2\right]+2\frac{1}{n^{1/2}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{Y}_{-1}(0)'\right\}+o_p(1), \qquad (16)
\end{aligned}$$

where the last equality holds by Assumption 2 and Lemma 12(a).

From (15) and (16), we have

$$\begin{aligned}
&\begin{bmatrix} t^{\#} \\ \sqrt{n}\left\{\frac{1}{nT^2}tr\left(\tilde{Y}_{-1}\tilde{Y}_{-1}'\right)-\frac{1}{6}\hat{\sigma}^2\right\} \end{bmatrix} \\
= \; & \sqrt{n}\begin{bmatrix} \frac{\sqrt{12}}{\sigma^2}\sqrt{n}\left[\frac{1}{nT}tr\left(\tilde{Y}_{-1}(0)\,\tilde{e}'\right)+\frac{1}{2}\sigma^2\right] \\ \frac{1}{nT^2}tr\left(\tilde{Y}_{-1}(0)\,\tilde{Y}_{-1}(0)'\right)-\frac{1}{6}\sigma^2 \end{bmatrix}+\begin{bmatrix} 0 \\ 2\frac{1}{n^{1/2}T^2}tr\left\{\left(\tilde{Y}_{-1}-\tilde{Y}_{-1}(0)\right)\tilde{Y}_{-1}(0)'\right\} \end{bmatrix}+o_p(1) \\
\Rightarrow \; & N\left(\begin{bmatrix} 0 \\ -\frac{\mu_\theta}{24}\sigma^2 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{45}\sigma^4 \end{bmatrix}\right),
\end{aligned}$$

where the last weak convergence holds by Lemma 9(a) and Lemma 12(b). ■

Part (b) holds by Lemma 9(c). ■

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Table 1. Size of tests

	$t^{\#}$					t^{+}					V_g					MPP				
$N =$	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250
$T =$																				
25	8.3	9.2	10.6	12.4	16.0	4.4	4.6	4.3	3.9	3.6	0.5	1.2	1.1	1.0	0.5	3.0	4.0	4.8	4.9	5.2
50	6.6	7.0	6.9	7.9	9.4	5.4	5.7	5.6	6.7	7.2	0.7	1.8	2.2	2.3	1.6	3.2	4.3	4.6	4.8	4.6
100	5.4	5.7	5.6	6.7	7.2	5.7	5.0	4.9	5.0	4.8	1.0	2.1	2.8	2.9	2.9	3.0	3.6	4.6	4.4	4.6
250	5.2	5.2	5.8	5.3	5.7	6.2	5.6	5.5	5.3	5.3	1.2	2.5	3.0	3.5	3.7	2.7	4.0	4.4	4.5	4.6

Table 2. Size-adjusted power
Local alternative: $\theta_i \sim U\,[0, 2]$

	$t^{\#}$					t^{+}					V_g					$MPP\,(c=1)$					Power envelope				
$N =$	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250
$T =$																									
25	4.2	4.7	4.8	5.1	5.4	8.8	9.9	10.6	10.9	10.8	10.7	12.0	12.7	13.0	13.2	12.7	14.0	15.7	16.0	16.1	8.1	13.1	15.9	17.1	18.3
50	3.9	4.2	4.5	4.9	5.1	9.3	10.2	10.2	10.8	11.3	11.2	11.9	12.6	12.9	13.8	12.8	13.7	15.6	16.4	17.1	7.0	12.9	15.9	17.1	17.7
100	3.8	4.0	4.4	4.7	4.8	9.6	9.9	10.6	10.6	11.6	11.2	11.1	12.7	13.7	12.7	12.0	13.6	14.4	15.4	15.8	7.0	12.4	15.5	16.9	18.2
250	3.5	3.8	4.2	4.5	4.6	9.2	10.1	10.5	10.8	11.5	10.9	12.0	12.4	13.3	13.7	13.2	14.6	14.5	15.3	15.7	6.6	12.5	15.4	18.1	18.6
Theory			5.0					12.0					14.3					17.4					20.4		

Table 3. Size-adjusted power
Local alternative: $\theta_i = 1\ \forall i$

	$t^{\#}$					t^{+}					V_g					$MPP\,(c=1)$					Power envelope				
$N =$	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250
$T =$																									
25	4.4	4.7	5.0	5.1	5.3	9.3	10.2	10.6	10.9	12.1	11.3	12.4	12.3	13.0	12.5	13.2	14.0	14.9	16.1	15.9	8.7	12.5	14.0	16.1	15.6
50	3.9	4.4	4.6	4.9	5.1	9.3	10.0	11.0	11.3	11.0	11.8	13.0	13.0	13.4	13.9	13.6	14.3	15.7	15.5	16.9	8.6	12.5	14.4	15.1	15.6
100	3.7	4.1	4.4	4.6	4.8	9.2	10.6	11.3	11.0	11.0	11.0	12.6	12.3	12.6	14.2	13.2	14.6	15.4	15.7	16.0	8.2	12.4	13.7	15.1	15.6
250	3.8	4.1	4.3	4.5	4.8	9.1	10.1	10.3	11.3	11.7	11.2	12.3	12.8	13.0	14.0	12.4	15.2	15.7	16.3	16.2	8.0	11.8	13.9	16.0	16.4
Theory			5.0					12.0					14.3					17.4					17.4		

Table 4. Size-adjusted power
Local alternative: $\theta_i \sim U\,[0, 8]$

	$t^{\#}$					t^{+}					V_g					MPP					Power envelope				
$N =$	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250
$T =$																									
25	1.8	2.6	3.0	4.2	5.1	25.5	32.4	38.0	42.8	47.1	46.0	53.5	60.0	63.3	65.8	51.7	62.9	71.3	76.2	80.2	91.9	95.2	95.5	96.0	95.9
50	1.2	1.8	2.2	2.8	3.9	25.2	31.9	36.1	43.0	46.6	47.0	54.0	58.3	62.9	69.3	51.6	62.0	70.6	76.8	82.5	90.6	93.3	94.5	94.6	95.4
100	1.0	1.5	2.0	2.6	3.4	24.9	32.4	36.2	42.0	48.3	47.6	52.4	60.4	64.2	65.9	50.1	62.9	70.7	75.5	81.1	89.7	92.2	93.7	93.8	94.4
250	0.8	1.3	1.9	2.1	2.9	24.0	31.5	37.0	41.6	47.2	45.3	54.7	59.6	64.4	69.9	52.6	65.2	70.9	76.1	80.0	88.6	91.6	93.1	93.8	94.1
Theory			5.0					59.3					74.7					88.2					94.8		

Table 5. Size-adjusted power
Local alternative: $\theta_i = 4 \; \forall i$

$N =$	$t^{\#}$					t^{+}					V_g					MPP					Power envelope				
	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250
$T =$																									
25	1.2	2.3	3.4	4.2	5.0	29.5	37.0	42.5	48.2	51.0	53.9	61.5	62.4	66.2	67.6	63.9	73.1	77.9	81.1	83.8	88.9	88.9	89.3	90.1	90.5
50	0.8	1.6	2.4	3.3	4.1	27.7	35.4	42.4	47.3	49.4	55.5	62.2	65.5	67.6	72.1	65.5	74.7	79.5	81.8	85.2	86.2	87.5	88.2	88.4	89.0
100	0.6	1.2	1.9	2.6	3.5	28.1	36.7	41.8	46.4	50.1	54.0	62.7	65.5	68.0	72.3	65.0	75.2	80.0	82.0	84.5	85.0	86.1	86.9	87.0	87.8
250	0.4	1.1	1.8	2.3	3.0	27.1	35.8	40.2	47.2	51.0	52.8	63.0	66.8	69.4	70.2	64.7	76.6	80.9	82.9	83.7	83.8	85.3	86.9	87.7	87.2
Theory			5.0					59.3					74.7					88.2					88.2		

Table 6. Size-adjusted power
Fixed heterogeneous alternative: $\rho_i \sim U\,[0.98, 1]$

$N =$	$t^{\#}$					t^{+}					V_g					MPP					Power envelope				
	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250
$T =$																									
25	4.5	4.4	4.6	4.7	5.1	7.8	11.9	15.8	26.3	46.8	10.3	12.8	19.1	31.1	67.7	12.2	14.3	23.1	38.5	82.5	4.7	19.4	37.1	65.3	95.8
50	3.2	2.9	2.7	2.4	1.9	12.6	20.0	32.2	56.3	87.6	15.6	31.1	51.0	74.6	99.9	16.9	38.2	62.6	85.4	99.9	19.9	58.8	87.9	99.2	100.0
100	1.6	0.8	0.7	0.3	0.1	17.8	44.9	61.8	92.6	99.9	41.3	76.3	93.8	99.6	100.0	47.6	84.5	96.7	99.9	100.0	72.8	98.5	100.0	100.0	100.0
250	0.8	0.0	0.0	0.0	0.0	37.5	90.3	98.9	100.0	100.0	77.9	99.6	100.0	100.0	100.0	78.7	99.7	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 7. Size-adjusted power
Fixed homogeneous alternative: $\rho_i = 0.99 \; \forall i$

$N =$	$t^{\#}$					t^{+}					V_g					MPP					Power envelope				
	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250	10	25	50	100	250
$T =$																									
25	4.5	4.5	4.7	5.1	5.4	8.8	11.9	16.6	26.9	50.7	9.8	14.2	21.4	35.5	67.6	10.5	16.9	28.0	46.9	83.2	6.4	17.4	31.1	53.8	89.9
50	3.0	2.9	2.7	2.6	2.1	11.3	22.2	37.7	60.1	93.0	17.0	34.9	56.2	83.8	99.7	19.9	42.5	69.0	93.9	100.0	18.9	49.9	79.4	97.5	100.0
100	0.9	0.7	0.4	0.3	0.1	21.4	46.9	73.8	96.2	100.0	38.9	80.1	98.2	100.0	100.0	48.7	90.0	99.7	100.0	100.0	64.3	96.5	100.0	100.0	100.0
250	0.0	0.0	0.0	0.0	0.0	54.2	93.6	99.9	100.0	100.0	97.6	100.0	100.0	100.0	100.0	98.2	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0